Pricing and Hedging of Guaranteed Minimum Benefits in Variable Annuities

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A thesis proposal submitted in partial fulfilment of the requirements for the degree of Bachelor of Commerce and Bachelor of Science (Honours in Actuarial Studies)
Declaration

I hereby declare that this submission is my own work and to the best of my knowledge it contains no material previously published or written by another person, nor material which to a substantial extent has been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis.

I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project’s design and conception or in style, presentation and linguistic expression is acknowledged.

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Abstract

Retirement and investment plans are becoming ever so important, especially after considering the shift from Defined Benefit to Defined Contribution superannuation schemes in the developed countries. Variable annuities with Guaranteed Minimum Benefits are gaining popularity in the superannuation market as products that can meet the demands of the ageing population. However, the annuity providers expose themselves to multiple risks lying within these variable annuities. In this thesis we provide a comprehensive pricing and hedging framework for various types of Guaranteed Minimum Benefits embedded in variable annuities under the regime-switching environment. We present insights into the pricing and hedging of these products, taking into account stochastic mortality, equity and interest rate risks. Our methodology is general enough to accommodate all Guaranteed Minimum Benefits that exhibit a fixed time horizon. This extends the current literature which has been primarily focused on pricing one particular product. We employ the Fourier Space Time-stepping algorithm and demonstrate its flexibility by applying it to both, pricing and hedging applications. Furthermore, it becomes increasingly important to account for all three risks (equity, interest-rate and mortality) when discussing hedging portfolios. We demonstrate the static hedging strategy and analyse its effectiveness through profit and loss distributions across time. The pricing and hedging analysis better quantifies the risks embedded in Guaranteed Minimum Benefits, which provides guidance to insurers and annuity providers.
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Chapter 1

Introduction

An annuity provides a specified income stream for a fixed or contingent period (such as the recipient’s whole life) in return for a stipulated premium paid either in prior instalments or in a single payment. There are two classes of annuities: fixed and variable. A variable annuity (VA) allows the policyholder to get exposure to the equity market by linking the level of payments to the performance of a chosen investment fund. This allows the individual to earn the additional equity risk premium on top of the mortality credit embedded in annuities. In comparison, fixed annuities do not allow the policyholder to participate in the equity market and thus, provide a more stable cash flow. Insurers often offer guarantees embedded in VAs (e.g. Guaranteed Minimum Benefits (GMBs)) to restrict the downside risk - making these products more appealing to potential annuitants as a hybrid between variable and fixed annuities. With the shift from Defined Benefit to Defined Contribution schemes in developed countries (e.g. US, UK, Japan and Australia), there is a growing demand in the ageing population for products that can manage their longevity risk, such as the VA with embedded GMBs. Furthermore, some countries (e.g. US and Canada) provide tax-shelters for the investment gains of VAs. This has led to the rapid growth in VA sales in the global markets; the rise in popularity was also accompanied by an annuities arms race in the early to mid 2000s (Grepin et al., 2009).

GMBs in VAs can be classified into the following sub-types:

- Guaranteed Minimum Maturity Benefit (GMMB) - A guarantee that provides the policyholder with a minimum benefit on maturity of the contract.
- Guaranteed Minimum Income Benefit (GMIB) - A guarantee that provides the policyholder with a minimum amount of income stream for a given period of time.
- Guaranteed Minimum Death Benefit (GMDB) - A guarantee that provides the policyholder’s beneficiaries with a minimum benefit in the event of death of the policyholder.
• Guaranteed Minimum Withdrawal Benefit (GMWB) - A guarantee that allows the policyholder to recoup at least the initial investment amount by continually withdrawing a portion of the entire investment. Any excess in the investment account is paid at maturity.

• Guaranteed Lifetime Withdrawal Benefit (GLWB) - A guarantee that allows the policyholder to continually withdraw until death, even after the investment account is depleted. After death, any excess in the investment account is paid to the beneficiaries.

This research primarily focuses on pricing and hedging of GMMBs, GMIBs, GMDBs and GMWBs. The GLWB, on the other hand, is a guarantee which lasts for life. This differs from the other guarantees as the insurer is uncertain of the maturity date for these contracts. Since GMMBs, GMIBs, GMDBs and GMWBs all have pre-specified maturity dates, they are more comparable amongst one another and therefore is the focus of this research.

1.1 Motivation

GMBs embedded in VAs have features of both mutual funds and traditional life annuities. Due to the long-term nature of these products and their suitability to an ageing population, it is necessary for insurers to incorporate stochastic mortality models when valuing variable annuities. In recent years, mortality risk has gained prominence as life expectancies have been improving faster than what was previously anticipated. Cairns et al. (2006) show that stochastic models are better than non-stochastic ones in capturing the mortality risk. These results motivate us to use a stochastic mortality model in this thesis to describe the mortality experience.

Moreover, given that the guarantees can be written in nominal amounts, we require a stochastic interest-rate model for the evolution of interest rates during the life of the contract. This is done so that we have a more realistic estimate of the discounting factor in valuing the long-term benefits a VA with GMBs would provide. Equitable Life UK is a typical example of a failure in accounting for interest rate risk. Equitable Life began selling life insurance policies in the 1950s with minimum pension payouts and a bonus when their policy matured, i.e. a combination of GMIBs and GMMBs. In the 1970s, it guaranteed very high interest rates due to the high inflation at the time. At the turn of the century, continual low inflation and interest rates meant that Equitable Life was unable to fund its commitments; by December 2000, it closed to new business and reduced payouts to its existing members. In hindsight, it is clear that a misunderstanding of the interest rate risk and poor risk management of GMBs had aggravated the situation.

By guaranteeing their policyholders a minimum level of return, the insurance providers also expose themselves to the equity risk. This risk manifests when the investment returns are lower than expected. This is particularly prominent during drastic financial events, such as the Global
Financial Crisis (GFC). Following the GFC in 2007-2008, many life insurers also experienced large losses due to their exposure to the equity markets through the variable annuity accounts (Grepin et al., 2009). Moreover, the interest rates were also lower during the GFC which makes the value of future guarantees more expensive than previously calculated. This motivates us to use a regime-switching Log-Normal model, which can allow for stochasticity in the financial market (stock return, volatility and interest rate).

Insurers also need to bear the full weight of these risks due to a lack of reinsurance markets. Reinsurers are reluctant to participate as there is little risk reduction from pooling as most of the risks are systematic in nature. Furthermore, custom options designed for the sole purpose of hedging GMBs are too expensive to be used for hedging purposes (Dellinger, 2006). Hence, insurers are forced to find new ways to effectively manage these risks. The existing literature often employs dynamic hedging strategies. However, this is an infeasible solution in practice as the total transaction cost over the long contract period is too high. Instead, more realistic strategies such as static and semi-static hedging are proposed in this research; thus there is a trade-off between transaction costs and hedge effectiveness. Our aim is to bring more attention to the risks and risk management methods involved with GMBs in VAs to the annuity provider.

Despite their popularity in the US and Japan, VA sales have largely been unsuccessful in Australia. The Australian annuities market remains weak and underdeveloped, with very few market players actively involved in VAs market (such as Challenger who offers the Guaranteed Income Fund\(^1\)). However, the Australian superannuation market has grown quickly since the introduction of compulsory superannuation in 1992 and is currently valued at $2.02 trillion as of June 2015 (ASFA, 2015). The recent Financial Systems Inquiry (FSI, 2014) recommends the inclusion of longevity protection products in the default option for superannuation, i.e. the Comprehensive Income Product for Retirement (CIPR). There is a growing demand for adequate pricing and risk management of retirement investments for Australians who are now accumulating higher super account balances than ever before. The overhead costs of Australian superannuation firms are around 1% of their total funds, much larger than the 0.1% average in Denmark (Frijters and Foster, 2015). The discrepancy can stem from a difference in understanding of the longevity and investment risks. Thus, we choose to calibrate our models with Australian data to shed more insight on the risk profile of GMBs for Australian insurers and super providers.

Lastly, there is not a lot of existing literature that discusses various types of GMBs together. Bauer et al. (2008) consider the pricing of GMMBs, GMIBs, GMDBs and GMWBs using the geometric Brownian motion (GBM) and a deterministic mortality model. Bacinello et al. (2011) value the same types of GMBs under stochastic interest rates and stochastic mortality. Our research will expand the literature in this area by pricing GMMBs, GMIBs, GMDBs and GMWBs under a regime-switching Log-Normal (RSLN) model and a stochastic mortality model. Hardy (2003) shows that the RSLN model better fits empirical data than the standard GBM. By using

a stochastic mortality model in conjunction with the RSLN model, we account for mortality, equity and interest rate risks. Moreover, we also compute the hedges for these GMBs and discuss the static and semi-static hedging strategies. The pricing and hedging of GMBs under the regime-switching environment are implemented through the Fourier Space Time-stepping (FST) algorithm, which has not been previously documented in the existing literature.

1.2 Research Aims

The aim of this research is to provide a comprehensive investigation on Guaranteed Minimum Benefits in Variable Annuities under stochastic mortality, equity risk and interest rate risk. This includes

- Developing a pricing framework for GMBs under the regime-switching Log-Normal model,
- Performing sensitivity analysis with respect to model parameters,
- Analysing static and semi-static hedging strategies and their effectiveness.

By addressing these points, we look to extend the existing literature in generalised pricing frameworks for VAs (see Bauer et al., 2008 and Bacinello et al., 2011). Furthermore, this research provides static hedging analysis under the regime-switching framework and tracks its dwindling effectiveness throughout time to justify the use of semi-static hedging strategies. We also aim to better quantify the risks embedded in GMBs, especially for Australian insurers and super providers.

1.3 Summary of Results

We conduct numerical experiments for pricing and hedging of various GMBs with 15-year maturities sold to 50-year-old Australian males (i.e. the contract lasts until age 65).

- The fair value of a GMMB increases for the first several years, before starting to decrease. Similar behaviour is also observed for the GMIB. The increase in value of guaranteed benefits with longer maturities compensates for the increasing uncertainties further along in the future. The price decrease is caused by the increasing mortality which discounts the benefit since GMMBs and GMIBs only provide benefits if the policyholder is alive.
- For a GMDB, the value is exceptionally low in the early years, in comparison to its living benefit counterparts such as GMMB and GMIB. However, when the length of the contract
increases to 40 years and beyond, its value increases quickly and exceeds that of GMMBs and GMIBs. Nevertheless, in the context of GMBs maturing at retirement age of 65, the value of the GMDB is almost negligible.

- Mortality has almost no effect on the pricing of GMWBs with shorter maturities (less than 30 years) under the static withdrawal case. This is due to the low probability of death in the early years. Although mortality does have an effect for GMWBs with longer maturities (more than 30 years), the value of the GMWB has diminished significantly and is close to zero, due to the low level of the guarantee.

- A static hedging portfolio provides limited effectiveness compared to an unhedged one. Its hedge effectiveness diminishes rapidly after some of its assets mature. Thus, a semi-static hedging portfolio is recommended in that case.

- A hedging portfolio made up of various GMMBs performs better than a hedging portfolio constructed of simpler instruments such as European calls, zero-coupon bonds and q-Forwards.

1.4 Thesis Outline

The structure of this thesis is as follows. Chapter 2 introduces the current literature surrounding GMBs. This includes pricing and hedging of guarantees. Furthermore, the existing literature of mortality models and regime-switching models is also discussed.

Chapter 3 details the methodology of this research. This chapter introduces the mortality and financial models used, and provides a general valuation framework for GMMBs, GMDBs, GMIBs and GMWBs. Additionally, static and semi-static hedging strategies are derived.

Chapter 4 consists of the pricing and hedging results. Price sensitivities to model parameters are performed and various profit and loss scenarios are plotted. The effectiveness over time of a static hedging portfolio is also analysed.

Chapter 5 concludes the thesis. This chapter provides an outline of the contributions of this research. It also discusses the scope and limitations of the research conducted and proposes possible future research.
Chapter 2

Literature Review

In this chapter, we provide the literature review which contains four streams of literature. We discuss the literature related to pricing and hedging of GMBs in Section 2.1. The current literature regarding stochastic mortality models is outlined in Section 2.2. In Section 2.3 we discuss the literature on regime-switching financial markets. Finally, in Section 2.4 we provide the literature on Fast-Fourier Transform (FFT) based methods for derivative pricing.

2.1 Current Literature on the pricing and hedging of GMBs

During the early years of VAs, insurers were only concerned with pricing the guarantees correctly. From the insurer’s perspective, the guarantees bear multiple risks, including mortality risk, equity risk and interest rate risk. Often, the GMBs pricing framework consists of using a financial model to capture the equity and interest rate risk, and a mortality model to capture the mortality risk. This is done with the reasonable assumption that the mortality process is independent of the financial markets (see Ulm, 2013; Fung et al., 2014; Da Fonseca and Ziveyi, 2015).

Aase and Persson (1994) provide an early attempt at pricing minimum guarantees. They examine traditional life insurance products (such as endowments and term insurances) and utilise no-arbitrage pricing and martingale theory to derive the optimal premiums. Since then, the pricing literature has grown considerably and more complex procedures have been developed to accommodate the relaxation of various underlying assumptions. There has been a shift from using deterministic models to stochastic models for modelling mortality evolution in the existing literature. Furthermore, researchers have been developing and using more generalised models. This is seen in the financial literature, where regime-switching models and Lévy processes provide a more general approach than the geometric Brownian motion (GBM) model (Hamilton, 1989; Cont and Tankov, 2003) and the mortality modelling literature, where the models have
advanced from discrete-time regression into continuous-time affine stochastic processes.

Unfortunately, there is no extensive discussion on hedging these guarantees in the early literature. Prior to the 1990s, companies often thought that the cost of extra benefits were insignificant and did not hold extra reserves (Coleman et al., 2007). Hyndman and Wenger (2013b) acknowledge that reinsurance was a viable risk management strategy for GMBs. However, the risk premiums continued to increase as reinsurers became more aware of the risks embedded in these guarantees. Moreover, after the GFC, reinsurers no longer offered coverage to GMWBs and GLWBs (Hyndman and Wenger, 2013a). Due to the increasing awareness of the risks associated with GMBs in VAs and the decreasing availability of reinsurance, insurers and superannuation providers aim to develop better internal risk management systems. In this research, we address the gap in literature by providing a novel approach for pricing and hedging of various GMBs under regime-switching models and affine stochastic mortality processes.

2.1.1 Comprehensive GMB Studies

There have not been many studies which discuss the various types of GMBs in conjunction. Bauer et al. (2008) provide a general pricing framework for various guarantees: GMMBs, GMDBs, GMIBs, and GMWBs. The authors provide a consistent and extensive analysis of these guarantees using the GBM model and actuarial life tables. They also assume that the policyholders have a predetermined strategy and act in an optimal manner. Bauer et al. (2008) find the risk-neutral values of these guarantees through a combination of Monte-Carlo methods and finite mesh discretization approaches proposed by Tanskanen and Lukkarinen (2004). Another general pricing framework is discussed in Bacinello et al. (2011), who propose stochastic mean-reverting processes for the equity dynamics and for the mortality rate. Bacinello et al. (2011) also uses an alternative computational method, the Least Squares Monte-Carlo method described in Bacinello et al. (2010), to evaluate the fair price to be charged.

In the following subsections (2.1.2 - 2.1.5), we describe the pricing literature review related to GMMBs, GMDBs, GMWBs, and GLWBs, respectively. The last subsection Section 2.1.6 provides an overview of the literature related to hedging of GMBs.

2.1.2 Valuation of GMMBs

The GMMB, also known as the Guaranteed Minimum Accumulation Benefit (GMAB), is one of the simplest guarantees in the class of GMBs. Conditional on the policyholder being alive at maturity, its payoff function is illustrated in Figure 2.1. Here, $b(s)$ denotes the benefit received at time $s$, given the fund value $F$, guarantee level $G$, and time of death for the policyholder $t_d$. 
For a GMMB,

\[ b(s) = \begin{cases} 
\max(F(s), G) & \text{if } s = t_T \text{ and } t_d > t_T, \\
0 & \text{otherwise.}
\end{cases} \]  \hspace{1cm} (2.1)

Brennan and Schwartz (1976) show that the benefit can be decomposed into a guaranteed amount \( G \) at maturity, plus a European call option whose strike is \( G \). Bauer et al. (2008) provides a simple pricing framework for GMMBs using the GBM and life tables to account for the financial and mortality components respectively. Since Bauer et al. (2008) do not account for systematic mortality risk, Bacinello et al. (2011) fill the gap in the literature by pricing the GMMB under stochastic mortality as well as stochastic interest rates and stochastic volatility. Furthermore, Krayzler et al. (2012) provide analytical pricing formulas for the GMMB with various additional riders under stochastic interest rates and stochastic mortality. The riders the authors have included in their analysis are capital protection, minimum interest return, ratchets and the maximum of the three mentioned.

Recently, there has been a surge of literature targeting the valuation of GMMBs written on several assets. In many of the papers in the existing literature, valuation methods are based on a single underlying asset, but this does not reflect the insurer’s practices (Hardy, 2003). Due to the complexities of multi-asset models, valuation of GMBs written on several assets primarily focus on the simpler GMBs (GMMBs and GMDBs). Ng and Li (2011) value the GMMB and GMDB written on multiple assets under a multivariate regime-switching model. The authors assume mortality rates evaluated through the Makeham law (Makeham, 1860) and calculate the value of these riders using Monte-Carlo simulations. Da Fonseca and Ziveyi (2015) values GMMBs and GMDBs written on two underlying assets whose dynamics are described by the affine type processes proposed in Heston (1993). Furthermore, the authors assume that the mortality evolves according to a one-factor affine process.
2.1.3 Valuation of GMDBs

There is a vast amount of literature pertaining to the valuation of GMDB riders. The GMDB guarantees a minimum benefit to the policyholder’s beneficiaries, if the policyholder dies during the term of the contract, as shown in Figure 2.2. In comparison to Equation (2.1), the benefit received $b(s)$ is now received at the time of death $t_d$:

$$
 b(s) = \begin{cases} 
 \max\{F(s), G\} & \text{if } s = t_d \text{ and } t_d \leq t_T, \\
 0 & \text{otherwise.} 
\end{cases}
$$

(2.2)

Milevsky and Posner (2001) provide an algorithm to compute the equilibrium fee to be charged for providing GMDBs. They assume that the VA account dynamics evolve according to the GBM framework. Furthermore, the authors utilise a deterministic mortality process which follows the Gompertz mortality law (Gompertz, 1825). Milevsky and Posner (2001) document that, in general, the insurance industry is overcharging policyholders for these guarantees. Milevsky and Salisbury (2001) notice that policyholders are likely to lapse their policy when the embedded option is deep out-of-the-money as there is no incentive to be continually charged a fee when the guarantee is unlikely to be exercised. This motivates the authors to calculate the additional surrender charge the insurer should place to discourage policyholders from early exits. Milevsky and Salisbury (2001) assume that policyholders act optimally and formulate the lapse decision as an optimal stopping problem. The “optimal” charge is achieved when the policyholder is indifferent between keeping and surrendering the policy before maturity. Milevsky and Salisbury (2001) price the GMDB using the GBM model for the investment fund and a constant hazard rate for mortality.

There are various features that can be added to the GMDB. One such feature is the option to change the level of exposure to the equity market by transferring funds between the market-varying account and the guaranteed-growth account. This means that the investment fund value $F(t)$ is separated into two, a risky stock and a guaranteed-growth account, and that the policyholder is able to alter the amounts in each account to change the level of risk he bears throughout the contract period. Ulm (2006) values these features using Milevsky and Salisbury’s (2001) setup, that is, a GBM model for the market dynamics and a constant force of
mortality. Ulm (2006) shows that the option to transfer to the guaranteed-growth account has no value and will not be exercised unless the fixed growth rate is smaller than the risk free rate less any insurer’s charges. Otherwise, the value of the option is to be calculated and the approximate location of the optimal exercise boundary can be determined. Ulm (2006) provides numerical solutions through a trinomial lattice approach described in Hull (1997). Gao and Ulm (2012) note that Ulm (2006) assumes that the VAs market is complete. Instead, they evaluate the optimal allocation of funds between the fixed and variable accounts using utility theory. The authors use a constant relative risk aversion (CRRA) utility for the insured and assume equal levels of risk aversion in the policyholders. Furthermore, they price the GMDB through maximizing the discounted expected utility of the policyholders and beneficiaries. Similarly to Ulm (2006), numerical solutions are provided using the trinomial lattice approach. Ulm (2013) suggests improvements to Ulm (2006) by incorporating a better model for mortality (using the Makeham law of mortality instead of a constant force of mortality), and provide a fast and accurate approach for solving the PDE derived in Milevsky and Salisbury (2001) using Laplace Transform techniques.

Although most of the existing literature has focused on the arbitrage-free pricing, it is debatable whether no-arbitrage theory is applicable as guaranteed benefits are not readily traded and there is no sizeable market for hedging mortality risk. Feng and Volkmer (2012) point out that risk measures of future gross liabilities can be used as pricing principles for rider charges. They provide analytical calculations to the GMDB and GMMB based on the GBM and life tables.

Another popular assumption employed when pricing GMBs is the independence between mortality and interest rates. There is a significant amount of research focusing on the development of models to describe the mortality component in GMBs. These include continuous-time models (see Dahl, 2004; Biffis and Milossovich, 2006 and Biffis, 2005) and discrete-time models (see Lee, 2000 and Lin and Cox, 2005). Gaillardetz (2008) notes that the pricing results obtained when using the actuarial present value are often inconsistent with those observed in the insurance market. Rather than using a mortality model, Gaillardetz (2008) utilises the observed premiums for term life insurances and pure endowments as a proxy to price the mortality factor in GMDBs. It appears reasonable as the insurer offering GMDBs is likely to issue standard insurance products as well. The author assumes that the GMDB is purely invested in a bond fund, which evolves under stochastic interest rates, and develops an age-dependent, mortality risk-adjusted martingale probability measure for such a contract. Hence, Gaillardetz (2008) proposes a way to value interest rate guarantees under stochastic interest rates and observed market premiums of term life insurances and pure endowments.
2.1.4 Valuation of GMWBs

GMWBs have been a major driving force behind VA sales. Yet, many academics have raised concerns that insurers have been consistently undercharging the fees related to GMWBs (see Milevsky and Salisbury, 2006; Dai et al., 2008 and Chen et al., 2008). Furthermore, these studies were conducted using pre-GFC data, thus implying that the guarantees were even more undervalued than previously thought. The policyholder’s withdrawal behaviour presents a major challenge to academics and insurers in correctly determining the fair price of GMWBs. Generally, the GMWB rider allows the policyholder to continually withdraw a pre-specified amount until the guarantee account value reaches zero. However, the policyholder pays an extra penalty fee if he or she withdraws more than the contract-specified amount. There are two types of withdrawal behaviour, either static or dynamic. Static withdrawal behaviour refers to constant withdrawals, either continuously or at discrete time points, while dynamic withdrawal behaviour suggests a withdrawal rate, \( \gamma(t) \), which varies with time. Let us denote the guarantee account value at time \( t \) as \( A(t) \). The policyholder receives the withdrawal amounts \( \gamma(t) \) at the withdrawal times \( t_i \) for \( i = 1, 2, \ldots, T - 1 \), and also depletes his guarantee account on either maturity or death, whichever comes first. The benefits received is depicted in Figures 2.3 and 2.4 and is expressed as:

\[
\begin{align*}
    b(s) &= \begin{cases} 
    \gamma(s) & \text{if } s = t_i \text{ and } t_i < \min(t_d, t_T), \\
    \max(A(s), F(s)) & \text{if } s = \min(t_d, t_T), \\
    0 & \text{otherwise}. 
    \end{cases} 
\end{align*}
\] (2.3)

Figure 2.3: VA embedded with a GMWB rider which is terminated in the event of policyholder dying before maturity of the contract.

Milevsky and Salisbury (2006) value the VA with a GMWB rider under static and dynamic withdrawal behaviours. They assume that the withdrawals occur continuously, there is no mortality, and that the investment fund evolves under the GBM. Under the static behaviour assumption, the authors demonstrate that the GMWB can be split into a Quanto Asian put option and a term-certain annuity. For the dynamic behaviour, the authors assume policyholders behave optimally and always act to maximise the annuity value. Hence, the valuation becomes an optimal stopping problem, equivalent to that encountered when valuing American put options. To solve
Figure 2.4: VA embedded with a GMWB rider in the case where there is policyholder lives to maturity of the contract.

this problem, the authors propose a finite difference algorithm using the penalty approximation approach, but do not provide any numerical results. Grossmann et al. (2007) outlines various types of finite difference methods which can be used to solve PDEs numerically.

Dai et al. (2008) extend the framework developed in Milevsky and Salisbury (2006) by providing a rigorous derivation of the dynamic withdrawal case using optimal stochastic control theory and derive the associated Hamilton-Jacobi-Bellman (HJB) equation. Dai et al. (2008) consider withdrawals at discrete time intervals and utilise an efficient finite difference algorithm to calculate the optimal withdrawal amounts at each point. The authors also show that the solution of the discrete withdrawal time model converges to that of the continuous case. Similar to Milevsky and Salisbury (2006), Dai et al. (2008) also ignore mortality effects, and use a GBM model for the underlying fund, with the reasoning that the influence of equity changes outweigh mortality and interest rate changes. Chen and Forsyth (2008) extend the discrete withdrawal setting in Dai et al. (2008) to pricing continuous withdrawal contracts.

Similar to Milevsky and Salisbury (2006) and Dai et al. (2008), Chen et al. (2008) also value GMWB contracts using no-arbitrage arguments and assume no mortality. The authors extend the previously used GBM modelling framework for underlying asset dynamics to include jumps. Chen et al. (2008) also investigate the effects of sub-optimal behaviour using the approach proposed in Ho et al. (2005). Chen et al. (2008) determine the effects of various parameters and assumptions on the fair fee charged on the guarantee. An interesting result is that reset provisions have no effect on the price of GMWBs. Furthermore, the authors argue that the mutual fund should reflect real-world practices, and charge a separate fee for managing the fund which was not discussed in Milevsky and Salisbury (2006) and Dai et al. (2008). Chen et al. (2008) provide results for discrete-time withdrawals and reinforce Dai et al.’s (2008) result of a discrete model converging to that of the continuous case through numerical tests. Moreover, it is shown that there is no significant change in the fair fee charged when the withdrawals periods change from yearly to monthly.

The common assumption of constant interest rates throughout the early literature for GMBs is unrealistic as VAs are long-dated contracts. Lin and Tan (2003) and Kijima and Wong (2007) consider pricing equity-linked annuities under stochastic interest rates. In terms of the GMWB
literature, Peng et al. (2012) price GMWBs under stochastic interest rates under static withdrawal assumptions. The authors find the lower bound via Roger-Shi’s method (Roger and Shi, 1995) and the upper bound via Thompson’s method (Thompson, 1999) to obtain an approximate solution to the price. Furthermore, Peng et al. (2012) analyse the impact of their model parameter values, in particular, the impact of stochastic interest rates on fees charged in GMWBs. They find that the value of the GMWB increases with increasing interest rate volatility, except for when the interest rate is negatively correlated to the stock, in which case, the value first decreases until a minimum value is reached, and then begins to increase again.

Hyndman and Wenger (2013b) consider fair pricing of the GMWB rider from the insurers’ and the policyholders’ perspectives. Motivated by Milevsky and Salisbury’s (2006) work, Hyndman and Wenger (2013b) prove the existence and uniqueness of a fair fee. The authors also decompose the value of the GMWB rider into two parts: the value of the guarantee under the no-lapse case, and the value of the option to lapse. The authors focus on the theoretical approach and outline numerical procedures in their subsequent research (see Hyndman and Wenger, 2013a).

Bacinello et al. (2014) utilise a general Lévy processes framework in their valuation of GMWBs with dynamic withdrawals. By using Lévy processes, the authors forego the normality assumption in the GBM framework, and obtain desirable properties such as capturing fat tails, skewness and kurtosis in asset returns. The authors use a dynamic programming algorithm to price the VAs embedded with GMWBs. The authors also remark that the same approach can also be applied to the static withdrawal case.

Another large obstacle arising in the academic literature dealing with GMWB valuation is the issue of efficiency in terms of computational time. This problem has restricted the early literature to consider only elementary deviations from the vanilla GMWB rider. Yang and Dai (2013) propose an accurate and flexible tree model that calculates the fair value of GMWBs. The authors show that mortality models can be incorporated into the tree and analyse the effect of mortality improvements on valuing GMWBs. Furthermore, the existing literature commonly considers GMWBs in non-deferred annuities, whereas in practice, most GMWBs are associated with deferred annuities. Yang and Dai (2013) show that the more popular deferred annuities would require significantly higher fees than immediate annuities. With a simple deterministic mortality model, the authors compare their numerical solutions to that of the Monte Carlo simulation and show that the tree procedure is numerically accurate and robust.

In response to the concern regarding high computational times, Luo and Shevchenko (2014) present an alternate computational algorithm. The algorithm relies on backward time-stepping between withdrawal dates. The authors use this approach to solve the optimal stochastic control problem for GMWB as outlined in Milevsky and Salisbury (2006). They use the pricing framework outlined in Dai et al. (2008) (a GBM for the underlying fund and do not model mortality) and show that the new algorithm provides significantly faster results that are as accurate as
those obtained by the finite difference method.

Another concern is that in a standard GMWB setup, the insurer deducts proportional fees from the policyholder’s account value. Hence, if the guarantee is in-the-money, then the account value is lower than the guaranteed amount and the fee charged is low. Similarly, if the guarantee is out-of-the-money, the fee is high. This setup increases the policyholders’ incentive to lapse their policies. State-dependent fees, however, are designed to reduce the policyholders’ incentives to lapse their policies. For example, Prudential UK introduced a variable annuity with minimum guarantees where the fee is deducted from the account only if the account value is below the guaranteed level (Bernard et al., 2013). However, this results in the insurer charging higher fees to recoup the lost earnings when the account value is above the guaranteed level. Bernard et al. (2013) study this new version of guarantees under a complete financial market setting and without mortality. They utilise a GBM to model the market and derive the equation for which the fair fee is to be solved. Delong (2014) extends Bernard et al.’s (2013) work, and model the market dynamics through a two-dimensional Lévy process to formulate the pricing problem under an incomplete financial market.

Liu (2010) demonstrates an alternative method to value the GMWB rider with discrete withdrawals under the Black-Scholes model. Rather than using the decomposition into an annuity-certain and Asian option as outlined in Milevsky and Salisbury (2006), Liu (2010) shows that the GMWB can also be represented as a put option whose maturity and payoff are random. The author shows that the two approaches are consistent through numerical results.

2.1.5 Valuation of GLWBs

The guaranteed lifetime withdrawal benefit (GLWBs) riders guarantee withdrawals until the end of the policyholder’s life. It can be likened to a GMWB rider without a maturity date, and the benefits received are as follows:

\[
b(s) = \begin{cases} 
\gamma(s) & \text{if } s = t_i \text{ and } t_i < t_d, \\
\max(A(s), F(s)) & \text{if } s = t_d, \\
0 & \text{otherwise.}
\end{cases}
\]

Holz et al. (2012) value GLWBs under different assumptions on policyholders’ behaviour and use the simplistic GBM to model the risky assets and life tables for the mortality experience. Kling et al. (2014) relax the constant volatility assumptions made in the GBM and value the GLWB rider with a stochastic volatility model for the underlying fund and life tables. In comparison to the gradual use of more generalised financial models, a stochastic mortality model in addition to the GBM for the financial market is considered in Piscopo and Haberman (2011). Fung et al. (2014) quantify the effect of systematic mortality risk and equity risk on the fee charged in
GLWBs. Like Piscopo and Haberman (2011), Fung et al. (2014) utilise a GBM in conjunction with a stochastic mortality process and assume static withdrawals. The authors focus on the interaction between equity risk and systematic mortality risk which is not examined in Piscopo and Haberman (2011). The authors also examine the effectiveness of a static mortality hedge by comparing profit and loss distributions for the hedged and unhedged portfolios.

2.1.6 Hedging

The importance of pricing the various guarantees has been emphasised in the existing literature on VAs. However, risk management requires equal or even greater attention. Hedging is one of the most powerful tools insurers can use for managing the risks in VAs. Unfortunately, the products which better suit the consumers’ needs are often more troublesome to hedge for insurers. There are some cases where insurers have neglected the risk management of VAs. A prevailing case is one of Equitable Life (2000). At its peak, Equitable Life had funds worth £26 billion under management. However, it allowed large unhedged liabilities to accumulate due to a large amount of guarantees sold on variable annuities. It found itself locked into offering high returns due to high inflation rates in the 1970s, and by December 2000, Equitable Life closed to new business. Though the losses of Equitable Life are not solely due to the large amount of guarantees sold, a large portion can be attributed to a misunderstanding of the risks embedded in their complex guarantees. In the recent years, there has been a considerable amount of research done on hedging guarantees in VAs. The general approach is to present the guarantee as an option and utilise the appropriate hedging techniques described in financial literature. There are two streams of hedging literature: one focuses upon hedging financial risks, and the second focusing on the hedging of longevity risk.

A common practice for insurers is to use dynamic delta-hedging to manage the risks embedded in the guarantees of VAs (Hardy, 2000). This approach poses several problems; theoretically, dynamic hedging assumes a continuous rebalance of the hedge portfolio in a manner such that its value is resistant to changes in market conditions. However, it is impossible to implement this method precisely in practice, especially due to transaction costs and liquidity constraints. This means that dynamic hedging strategies are likely to be not the optimal choice, after taking into consideration the transaction costs.
In contrast to dynamic hedging, static hedging is the approach of holding the hedged portfolio until maturity without any rebalancing. In the context of variable annuities, static hedging offers some advantages over dynamic hedging. Static hedging does not require rebalancing nor rely on particular model dynamics. Hence, it minimises transaction costs, which can be quite substantial, especially due to the long maturities of variable annuities. Static hedging for European options is discussed in Carr and Wu (2004). However, there is a challenge of hedging GMWBs due to their path-dependence. Kolkiewicz and Liu (2012) consider hedging path-dependent options by extending the local risk-minimisation method proposed in Föllmer and Schweizer (1989). Kolkiewicz and Liu (2012) also propose semi-static hedging strategies to lower the hedging error generated in static hedging portfolios. Kolkiewicz and Liu (2012) notice that the hedging error is also reduced, but never completely removed, by adding more options to the hedging portfolio. Semi-static hedging for GMWBs is shown to outperform dynamic hedging when the underlying fund evolves under a jump diffusion process (Liu, 2010). Moreover, Liu (2010) also discusses semi-static hedging of GMWBs under a stochastic volatility model, but do not compare its effectiveness to the dynamic hedging strategy.

When modelling VAs, due to their long term nature, it is more appropriate to incorporate one or more of the following components into the financial model - stochastic interest rates, stochastic volatility and jump behaviour. However, incorporating stochastic volatility and/or jump diffusion leads to an incomplete market. Moreover, when the market is incomplete, the delta-hedging strategy devised under a particular risk-neutral probability setting may no longer be optimal under the real-world dynamics.

Coleman et al. (2006) provide insights on hedging the financial risks in GMDBs under an incomplete market setting. For simplicity, the authors do not consider hedging the systematic longevity risk and also assume that any unsystematic mortality risk is diversified via the Law of Large Numbers when a large number of policies are sold. Coleman et al. (2006) demonstrate a discrete local risk minimisation hedging strategy on GMDBs under a stochastic interest rate model and a jump diffusion model. Under local risk minimisation, the insurer chooses an optimal hedging strategy that matches the final value of the option and minimises the intermediate cashflows required for rebalancing. The authors establish that hedging using standard options can significantly outperform hedging using futures of the underlying fund. A similar observation is made in Coleman et al. (2007), who investigate a local risk minimisation strategy for hedging GMDBs under stochastic volatility.

An alternative hedging criteria is quadratic risk minimisation. There is a substantial amount of financial literature in this area (see Föllmer and Schweizer, 1989 and Heath et al., 2001). Møller (1998), Lin and Tan (2003) use the quadratic risk minimising strategy to compute hedging strategies for equity-indexed annuities (EIAs) in a complete financial market. Both papers assume that the unsystematic mortality risk can be completely diversified away and also do not
model the systematic mortality risk. In addition, Lin and Tan (2003) incorporate stochastic interest rates in their modelling process. Due to the similarity between EIAs and VAs, the procedures and conclusions described in Lin and Tan (2003) can be easily adopted to suit VAs. In the incomplete market setting, Delong (2014) derives the optimal hedging strategy for GMWBs with state-dependent fees under the quadratic risk minimisation.

Systematic mortality risk arises from the uncertainty in the underlying mortality intensity, such as longevity improvements in the whole population. Hence, systematic mortality risk cannot be diversified away by increasing the pool of participants. Mortality derivatives proposed in the literature should be factored in to better manage the mortality risk (Blake et al., 2006). Cairns et al. (2006) remark that if mortality risk was readily traded through such derivatives, then the pricing and hedging techniques traditionally used for financial risks could also be applied to hedge mortality risk. Ziveyi et al. (2013) price options written on pure endowments and life annuities under the assumption that a mortality risk market exists. Blackburn et al. (2014) then utilise the pricing framework outlined in Ziveyi et al. (2013) to demonstrate dynamic hedging of pure endowment portfolios under a stochastic mortality and a stochastic interest rate model. They also compute the hedging errors that result from rebalancing the portfolio at discrete time points and note that the hedge error diminishes as the portfolio rebalancing frequency increases.

Mortality risk can also be naturally hedged as shown in Cox and Lin (2007), who construct a portfolio of life insurance policies and annuities. The systematic decrease in mortality rates creates an increase in the expected liabilities arising from annuities sold; however, the increase in liabilities is balanced by a decrease in the expected liabilities in life insurance policies. The authors emphasise the importance of natural hedging, and highlight that this should be of priority when designing life insurance products. Furthermore, Luciano et al. (2011) perform delta-gamma hedging for annuities and term assurances under mortality risk and interest rate risk. They reinforce the findings in Cox and Lin (2007) through studying the natural hedging opportunities between annuities and term assurances. Other research that considers hedging mortality risk include Dahl and Møller (2006), who hedge mortality risk through risk-minimizing strategies, and Fung et al. (2014), who examine static hedging of systematic mortality risk in GLWBs. The effectiveness of the static hedging is analysed through the difference in capital reserves required and profit and loss distributions under two scenarios - no hedging and static hedging of mortality risk.

2.2 Mortality models

Mortality modelling is a staple in actuarial science, with stochastic mortality models gaining popularity over the traditional deterministic mortality models. Gompertz (1825) suggests a
deterministic mortality hazard rate $h(x)$ which increases exponentially with age $x$:

$$h(x) = \alpha e^{\beta x}, \quad \text{(2.5)}$$

where $\alpha$ describes the baseline mortality rate and $\beta$ is the age factor. Although it fits well for the bulk of the population, the Gompertz model does not fit well to higher and younger ages. Makeham (1860) suggests adding an extra constant term $\lambda$ to improve the fit for younger ages, i.e. when $x$ is small:

$$h(x) = \alpha e^{\beta x} + \lambda. \quad \text{(2.6)}$$

However, it still does not capture the pattern in the young ages well: for example, the model cannot capture a spike in mortality for babies in the first year. Lee and Miller (2001) also note that the construction of model and parameter estimates may be significantly influenced by experts’ personal judgements.

It is also apparent that there are mortality improvements in the population over time and that the deterministic models consistently overestimate the mortality rates for advanced ages. Moreover, the rate of improvement seems to be stochastic in nature. These unanticipated improvements are much more significant at higher ages, and is likely to cause life insurers to incur losses in their businesses. Hence, insurers need models which can account for the mortality risk, i.e. the difference between the assumed mortality rate and the actual mortality experience. There have been multiple efforts to create a parsimonious model that can fit the current mortality trends as well as describe the longevity improvement through time. Since deterministic models fail to incorporate systematic changes in mortality, stochastic processes are used instead. These models form the basis of stochastic mortality modelling. In subsections 2.2.1 and 2.2.2, we describe two broad classes of stochastic models: time series models and continuous affine models.

### 2.2.1 Time Series Models

#### 1. Lee-Carter Model

Lee and Carter (1992) introduce a new discrete-time model for mortality, commonly known as the Lee-Carter (LC) model. Its main contribution is that it incorporates future mortality improvements into its mortality estimates. Cairns et al. (2008) note that this is the first and most popular stochastic model for mortality. It performs well for both, in-sample fits and mortality rates forecasts in the past two decades (Lee and Carter, 1992 and Renshaw and Haberman, 2006). Li and Lee (2005) also note that the model has succeeded in reducing the influence of subjective bias on mortality modelling.
The LC model is constructed as follows:

\[
\ln(m(t; x)) = \beta^{(1)}(x) + \beta^{(2)}(x) \times k(t) + \varepsilon(t; x), \tag{2.7}
\]

where \(m(t; x)\) is the central death rate for aged \(x\) in year \(t\), \(\beta^{(1)}(x)\) and \(\beta^{(2)}(x)\) are some age-dependent constants, \(k(t)\) is the time-varying index of the mortality level and \(\varepsilon(t; x)\) is an error term with zero mean and variance \(\sigma^2\), which includes some of the historical variations not captured by the model. A time series model is used to forecast \(k(t)\) and Lee and Carter (1992) find that the random walk with drift, ARIMA(0,1,0), fits best to their data. Although other ARIMA models can be used, the ARIMA(0,1,0) is by far the most popular specification in the existing literature. Under the random walk with drift, the model for \(k(t)\) becomes

\[
k(t) = k(t - 1) + \theta + z_t, \tag{2.8}
\]

where \(\theta\) is a drift parameter and \(z_t\) is a normally distributed random variable with zero mean and constant variance \(\sigma^2\). Note that this model is not uniquely defined yet, hence there are some constraints:

\[
\sum_{x=1}^{w} \beta^{(2)}(x) = 1 \quad \text{and} \quad \sum_{t=1}^{n} k(t) = 0, \tag{2.9}
\]

where \(w\) is the maximum attainable age and \(n\) is the final year the model describes. Under these constraints, it is easy to see that \(\beta^{(1)}(x)\) are the averages over time of \(\ln(m(t; x))\) (after applying the sum to both sides of Equation (2.7)). Lee and Carter (1992) use the singular value decomposition (SVD) method to find a least squares solution. A key advantage of the LC model is that death rates are strictly non-negative.

However, the LC model has some shortcomings. In the cases of limited data, the model does not perform well. It also does not always capture the age-cohort influences in the empirical data well (Renshaw and Haberman, 2003a and Li and Lee, 2005). Lee (2000) indicates the possibility of extensions over complicating the LC model without truly representing the fundamental aspects of mortality. Nevertheless, many extensions have been proposed in the literature (see Renshaw and Haberman, 2003b; Renshaw and Haberman, 2006; Lee and Miller, 2001). Another disadvantage is the large number of parameters needing to be estimated since the LC model does not make any parametric assumptions on the age effects (\(\beta^{(1)}, \beta^{(2)}\)). Furthermore, the survival probability cannot be expressed as a closed-form function of the variables of interest since it is a double exponential. This motivates us to use the Affine Term Structure Models (ATSM) rather than the LC model for pricing and hedging mortality derivatives such as GMBs.
2. Cairns-Blake-Dowd Model

Another famous mortality model is the two-factor Cairns-Blake-Dowd (CBD) model proposed in Cairns et al. (2006). It introduces two factors in a logit model; the first factor affects mortality-rate dynamics in an age-independent manner, whereas the second factor affects mortality-rate dynamics at higher ages more than at lower ages. The CBD model is constructed as follows:

\[
\ln \left( \frac{q(t; x)}{1 - q(t; x)} \right) = k^{(1)}(t) + k^{(2)}(t)(x - \bar{x}) + \varepsilon(t; x),
\]

(2.10)

where \(q(t; x)\) is the probability that an individual aged \(x\) at time \(t\) will die between \(t\) and \(t+1\), \(k^{(1)}(t)\) and \(k^{(2)}(t)\) are time-varying parameters, \(\bar{x}\) is the sample mean age and \(\varepsilon(t; x)\) is the residual with zero mean and variance \(\sigma^2_x\). Furthermore, \(k^{(1)}\) and \(k^{(2)}\) are forecasted through a two-dimensional random walk with drift. Let \(k(t) = (k^{(1)}(t), k^{(2)}(t))^\top\) denote the vector of parameters, then

\[
k(t) = k(t - 1) + \theta + CZ(t - 1),
\]

(2.11)

where \(\theta = (\theta^{(1)}(t), \theta^{(2)}(t))^\top\) is a \(2 \times 1\) vector representing the constant drift, \(C\) is a constant \(2 \times 2\) upper triangular matrix and \(Z\) is a two-dimensional standard normal variable. Cairns et al. (2006) find that \(\theta^{(1)}(t)\) is negative, which indicates that the general longevity of the population is improving over time. Furthermore, they also find that \(\theta^{(2)}(t)\) is positive, meaning that there is a lower mortality improvement at the higher ages. However, this leads to an undesirable property of the model, where given a high enough age, the model predicts a decrease in mortality improvement. Thus, the model fits well for ages between 60-90 but decreasingly so for any higher ages.

2.2.2 Affine Models

1. Dahl One-factor Model

Dahl (2004) proposes the use of affine models to capture the mortality evolution. Affine stochastic processes have already been proven to be effective in modelling financial instruments, which include, modelling interest rates, stochastic volatility for equity prices and modelling the default risk of corporate bonds (Biffis, 2005). These models are useful due to their tractability. In mortality modelling, affine models provide closed form solutions for survival probabilities and the continuous structure is useful in pricing and risk management applications. Dahl (2004), Dahl and Møller (2006) and Dahl et al. (2008) model
the mortality intensity, \( \mu(t; x) \), as a time-inhomogenous square-root process:

\[
d\mu(t; x) = (\rho(t; x) - \delta(t; x)\mu(t; x))dt + \sigma(t; x)\sqrt{\mu(t; x)}dW(t),
\]

(2.12)

where \( W(t) \) is a standard Brownian motion; \( \delta(t; x) \) is the mean-reversion coefficient of \( \mu(t; x) \) with mean \( \rho(t; x) \); \( \sigma(t; x)\sqrt{\mu(t; x)} \) is the diffusion coefficient. The time-\( t \) expected probability that an individual aged \( x \) will survive until time \( T \) is expressed in closed-form as:

\[
S(t, T; x) = e^{A(t, T; x) - B(t, T; x)\mu(t; x)},
\]

(2.13)

where \( A(\cdot) \) and \( B(\cdot) \) are solutions of the following ordinary differential equations (ODEs)

\[
\begin{align*}
\frac{\partial}{\partial t}B(t, T; x) &= \delta(t; x)B(t, T; x) + \frac{1}{2}\sigma(t; x)^2B(t, T; x)^2 - 1, \quad (2.14) \\
\frac{\partial}{\partial t}A(t, T; x) &= \rho(t; x)B(t, T; x). \quad (2.15)
\end{align*}
\]

The ODEs are then solved with the terminal conditions:

\[
A(T, T, x) = B(T, T, x) = 0. \quad (2.16)
\]

Over the years, we observe rectangularisation in the mortality data, that is, when the shape of the survival curve is becoming more rectangular. This phenomenon implies that the age of death is less dispersed, and a steeper increase in mortality is seen for higher ages. Luciano and Vigna (2005) observe that single-factor mean-reverting affine processes, such as the one proposed in Dahl (2004), are unable to capture the rectangularisation phenomenon and produce unrealistic survival probabilities at advanced ages. Luciano and Vigna (2005) show that non-mean reverting processes are better in describing mortality intensities, especially in the older population. Many other affine models (with more than one factor) have since been proposed in the literature such as the two-factor model by Biffis (2005), the generalised multiple factor model, the \( M \)-factor model (\( M \) denotes the number of factors) by Schrager (2006) and the three-factor model by Blackburn and Sherris (2013).

2. Biffis Two-factor Model

Biffis (2005) proposes a two-factor square-root diffusion model for the mortality intensity:

\[
\begin{align*}
d\mu(t) &= \rho_1(\bar{\mu}(t) - \mu(t))dt + \sigma_1\sqrt{\mu(t)}dW_1(t), \quad (2.17) \\
d\bar{\mu}(t) &= \rho_2(m(t) - \bar{\mu}(t))dt + \sigma_2\sqrt{\mu(t) - m^*(t)}dW_2(t), \quad (2.18)
\end{align*}
\]
where \( W = (W_1, W_2)' \) is a vector of standard Brownian motion processes; \( \sigma_1 \sqrt{\mu(t)} \) and \( \sigma_2 \sqrt{\bar{\mu}(t) - m^*(t)} \) are diffusion coefficients; \( \rho_1, \rho_2 > 0 \) are mean reversion coefficients of \( \mu \) to \( \bar{\mu} \) and of \( \bar{\mu} \) to \( m; \) \( m(\cdot) \) is chosen as a suitable demographic basis, such as the mortality table which acts as a time-varying target for the stochastic drift \( \bar{\mu} \) of \( \mu \). We can interpret \( m^* \) as the lower bound of the stochastic drift \( \bar{\mu} \), in other words, the asymptotic intensity.

Ziveyi et al. (2013) utilise the one-factor version of Equations (2.17) - (2.18) to price mortality derivatives, that is:

\[
d\mu(t; x) = \rho_1 (m(t) - \mu(t; x))dt + \sigma \sqrt{\mu(t; x)}dW(t), \tag{2.19}
\]

where \( \rho_1 \) is a mean reversion coefficient; \( m(\cdot) \) is once again chosen as a suitable basis; \( W \) is a standard Brownian motion and \( \sigma \sqrt{\mu(t; x)} \) is the diffusion coefficient. Ziveyi et al. (2013) demonstrate the tractability of affine models by providing an explicit pricing expression for European options written on deferred annuities given that the interest rate dynamics and the mortality dynamics are both specified via affine processes.

3. Schrager M-factor Model

By placing affine models in a multivariate setting, Schrager (2006) proposes that the general form of mortality intensity, \( \mu(t; x) \), of a person age \( x \) at time \( t \) is given by

\[
\mu(t; x) = g_0(x) + \sum_{i=1}^{M} Y_i(t)g_i(x), \tag{2.20}
\]

where \( g_i(x) \) are some positive functions and \( Y_1(t), ..., Y_M(t) \) are factors driving the mortality intensity. We represent all the factors in a single \( M \times 1 \) vector, \( \mathbf{Y} \), and the \( M \)-dimensional factor dynamics of \( \mathbf{Y} \) are given by the following diffusion process:

\[
d\mathbf{Y}(t) = \mathbf{A}(\theta - \mathbf{Y}(t))dt + \mathbf{\Sigma} \sqrt{\mathbf{V}(t)}d\mathbf{W}(t), \quad \mathbf{Y}(0) = \bar{\mathbf{Y}}, \tag{2.21}
\]

where \( \mathbf{W}(t) \) is an \( M \)-dimensional Brownian motion; \( \mathbf{A} \) and \( \mathbf{\Sigma} \) are \( M \times M \) matrices, \( \mathbf{\theta} \) is an \( M \times 1 \) vector; \( \mathbf{V}(t) \) is a \( M \times M \) diagonal matrix with the diffusion coefficients of the factors on the diagonal. This is known as an \( M \)-factor affine mortality model since the mortality intensity is an affine function with \( M \) factors. Schrager (2006) prices options embedded in insurance contracts under the proposed \( M \)-factor affine model and assumes a Gaussian model for the interest rate structure.

4. Blackburn and Sherris 3-factor Model
Blackburn and Sherris (2013) utilise multi-factor affine mortality models for fitting mortality rates. By having a multiple-factor model, the authors introduce both, mean reverting and non-mean reverting factors, which allow a deeper understanding of the significance of each factor across the age dimension. The instantaneous mortality intensity $\mu(t)$ is expressed as:

$$\mu(t) = \sum_{i=1}^{3} \zeta_i(t)$$

(2.22)

where $\zeta_i(t)$ are the factors, described by

$$\zeta(t) = (\zeta_1(t), \zeta_2(t), \zeta_3(t))^t,$$

(2.23)

$$d\zeta(t) = -\Delta \zeta(t)dt + \Sigma dW(t).$$

(2.24)

Here, $\Delta$ is a $3 \times 3$ matrix; $\Sigma$ is diagonal $3 \times 3$ matrix; $W(t)$ is a vector consisting of independent Brownian motions. Blackburn and Sherris (2013) fit the model to Swedish mortality data and note that a three-factor independent model provides a particularly good fit, especially towards older ages. Under the independence assumption, Equation (2.24) can be rewritten as

$$d\zeta_1(t) = -\delta_1 \zeta_1(t)dt + \sigma_1 dW_1(t),$$

(2.25)

$$d\zeta_2(t) = -\delta_2 \zeta_2(t)dt + \sigma_2 dW_2(t),$$

(2.26)

$$d\zeta_3(t) = -\delta_3 \zeta_3(t)dt + \sigma_3 dW_3(t),$$

(2.27)

where $\delta_1, \delta_2, \delta_3$ and $\sigma_1, \sigma_2, \sigma_3$ are the drift and volatility parameters of the processes; and $W_1, W_2, W_3$ are independent Brownian motions.

### 2.3 Regime-Switching models for the financial market

The traditional option pricing framework is to apply the geometric Brownian motion with constant volatility and interest rates to model the financial market. Options can then be priced using the Black-Scholes formula (Black and Scholes, 1973). However, empirical observations of financial data suggest that this modelling methodology is insufficient. The assumption of constant volatility and constant interest rates are questionable and many extensions have been proposed in the financial literature - stochastic volatility (Heston, 1993), stochastic interest rates (Vasicek, 1977; Cox et al., 1985; Hull and White, 1990) and jump terms (Merton, 1976). These models provide a better fit to empirical data than the GBM (Merton, 1996).

One approach for incorporating stochasticity in the volatility and interest rates is to assume that
these variables are controlled by a regime-switching process. Introduced by Hamilton (1989), regime-switching models are one of the most popular and practically useful types of models in econometrics and finance. Initially, Hamilton (1989) describes the regime-switching model with time series models for US postwar data on real Gross National Product (GNP) as

\[
y_t = n_t + \tilde{z}_t \\
n_t = \alpha_1 s_t + \alpha_0 + n_{t-1},
\]

where \( y_t \) is the log-GNP; \( \tilde{z}_t \) is an ARIMA(4,1,0) process; and \( s_t \) are states in a Markov chain such that the probabilities of moving from state \( s_{t-1} \) at time \( t-1 \) to another state \( s_t \) at time \( t \) are given by

\[
Pr(s_t = 1 | s_{t-1} = 1) = p, \\
Pr(s_t = 0 | s_{t-1} = 1) = 1 - p, \\
Pr(s_t = 0 | s_{t-1} = 0) = q, \\
Pr(s_t = 1 | s_{t-1} = 0) = 1 - q.
\]

Essentially, a regime-switching model combines a set of processes together and chooses one of them at any given time through a Markov chain. This is done by assuming that there exists a discrete Markov chain whose process dictates switches among a finite set of scenarios, or “regimes”, with each regime characterised by a set of parameters which are universal across the set of possible models. It is also known as the Markov Modulated model (Zhou and Yin, 2003). The regimes can represent various states of the economy; the switches can be interpreted as structural changes in an economy, financial crises, or political events. Hardy (2003) demonstrates that the regime-switching Log-Normal (RSLN) model fit stock returns better than the standard GBM model. The literature for RSLN models can be divided into two streams. The first stream deals with single regime-switching models and the second stream deals with double regime-switching models. The difference is that the double regime-switching model incorporates an extra jump risk in the share price during a regime change. For example, on Black Monday, the 19th of October 1987, the Dow Jones Industrial Average dropped by 22.61%, which motivates the usage of jumps in share price that can concurrently occur during a regime switch.

Naik (1993) proposes a double regime-switching model for pricing European option. Yuen and Yang (2010) extend Naik’s (1993) framework to numerically value European, American and other exotic options through a trinomial tree method. In contrast to Yuen and Yang’s (2010) numerical algorithm, Shen et al. (2014) utilise Fourier transforms to obtain the analytical pricing formula for European options. The authors show that the added premium due to regime-switching risk in double-regime switching models is significant. Shen et al. (2014) also compare the double regime-switching model against the single-regime switching model and the GBM using empirical data. They note that the double regime-switching model outperforms the other two models in both in-sample fit and out-of-sample prediction.

2.4 Fourier Space Time-stepping

The Fast Fourier Transform (FFT) is used in a variety of physical science applications, such as signal processing and image compression through its main advantage of providing a fast and efficient computational method of transforming values from the real space to the Fourier space. We outline one of the more popular FFT algorithms, the Cooley-Tukey algorithm in a later section (Equation (3.52)). Carr and Madan (1999) first popularise the use of FFT in option pricing by using a damped option price method. Due to the similarity between GMMBs and European options, Da Fonseca and Ziveyi (2015) extend its usage and value GMMBs written on several assets. It is important to note that the damped option price method can break down for deep out-of-the-money options where it starts to provide negative prices (Carr and Madan, 2009).

The usage of FFT algorithms is also prevalent in regime-switching models, such as Liu et al. (2006), who outline its uses in a single regime-switching model. The authors show mathematically and through numerical experiments that the solutions obtained via the FFT approximate the true value with sufficient accuracy. Shen et al. (2014) use the FFT to price European options under a double-regime switching model.

Another computational algorithm which utilises the FFT is the Fourier Time-Stepping (FST) algorithm as outlined in Jackson et al. (2008). Jackson et al. (2008) emphasise its versatility in pricing path-independent and path-dependent options. Furthermore, Surkov and Davison (2010) broaden the usage of the FST algorithm by demonstrating its applicability when computing the Greeks of options for hedging purposes. Lippa (2013) prices GMWBs under the GBM using the FST algorithm, and demonstrates that its numerical results are consistent with those documented in Chen et al. (2008). To our knowledge, Lippa (2013) is the first and only research which has utilised the FST algorithm for pricing GMBs. This thesis marks the second yet, portraying a more sophisticated pricing framework of GMBs under the regime-switching environment. This thesis also pioneers the usage of the FST algorithm in computing the Greeks of GMBs.
Chapter 3

Methodology

3.1 Mortality Model

There are various types of mortality models that are utilised in current actuarial literature (see Section 2.2). We adopt the two-factor independent version of Blackburn and Sherris’ (2013) setup for modelling mortality intensity. The model is chosen for its many attractive characteristics such as:

- The model is described in continuous-time;\(^1\)
- The force of mortality is non-negative;
- The model is tractable in the sense that it allows closed-form expressions for the survival probability;
- The model effectively captures mortality evolutions across all ages, including advanced ages.

The first three points are useful when pricing mortality linked derivatives such as the GMBs. Furthermore, Luciano and Vigna (2005) point out that not all ATSMs are able to capture the mortality evolution at advanced ages. In particular, one-factor ATSMs often fail in this regard. It is essential that the mortality model we choose is able to do so due to the long term nature of GMBs. Hence, the two-factor mortality model described in Blackburn and Sherris (2013) has been adopted. The rest of this section describes the construction of the mortality model and provides preliminary fitting to Australian male population data extracted from the Human

\(^1\)A discrete-time mortality model is also acceptable, as in practice, insurers often conduct their business in discrete time periods. However, the continuous model is desirable as it allows for meaningful results when deriving the mortality hedges.
Mortality Database.

Let $\mathcal{T}$ denote the time index $[0, T]$, where $T < \infty$. We assume a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F} := \{\mathcal{F}(t) | t \in \mathcal{T}\}$ satisfying the conditions of right-continuity\(^2\), and $\mathbb{P}$-completeness; $\Omega$ represents the complete set of possible events; $\mathbb{P}$ represents the real-world probability measure. We also assume that all sources of randomness are defined in this probability space, including a standard Brownian motion and a Markov chain.

In order to use the mortality model for pricing and hedging purposes, the model needs to be described under the risk-neutral measure $\mathbb{Q}$. Blackburn and Sherris (2013) show that there exists an equivalent martingale measure $\mathbb{Q}$ such that $S(t, T)$, the time-$t$ expected probability of an individual who is alive throughout $[t, T]$, can be represented as

$$
S(t, T) = \mathbb{E}_t^\mathbb{Q} \left[ e^{-\int_t^T \mu(s) ds} | \mathcal{F}(t) \right],
$$

where $\mu(s) = \mu^\mathbb{P}(s) = \mu^\mathbb{Q}(s)$ with $\mu^\mathbb{P}(s)$ and $\mu^\mathbb{Q}(s)$ denoting the instantaneous mortality intensities under the $\mathbb{P}$ and $\mathbb{Q}$ measures, respectively. We do not price unsystematic risk and assume that we are modelling population mortality rates since the insurer has a sufficiently large pool of policyholders to diversify away all the unsystematic mortality risk. The instantaneous mortality intensity, $\mu(t)$, is of an affine form and is described by

$$
\mu(t) = \zeta_1(t) + \zeta_2(t),
$$

where $\zeta_1(t)$ and $\zeta_2(t)$ are some latent factors given by

$$
d\zeta_1(t) = -\delta_1 \zeta_1(t) dt + \rho_1 dZ_1^\mathbb{Q}(t),
$$

$$
d\zeta_2(t) = -\delta_2 \zeta_2(t) dt + \rho_2 dZ_2^\mathbb{Q}(t).
$$

In Equations (3.3) and (3.4), $\delta_1, \delta_2, \rho_1,$ and $\rho_2$ are time-independent coefficients; $Z_1^\mathbb{Q}$ and $Z_2^\mathbb{Q}$ are independent Brownian motions under the $\mathbb{Q}$-measure. Furthermore, Duffie and Kan (1996) show that the general solution of Equation (3.1) is given by

$$
S(t, T) = e^{-C_1(t, T) \zeta_1(t) - C_2(t, T) \zeta_2(t) + D(t, T)},
$$

\(^2\)A filtration satisfies right continuity if $\mathcal{F}(t^+) = \mathcal{F}(t)$, where $t^+$ denotes the time right after time $t$.
where $C_1(t, T)$, $C_2(t, T)$ and $D(t, T)$ are solutions to the following set of ordinary differential equations (ODEs):

\[
\frac{dC_1}{dt} = 1 - \delta_1 C_1(t, T), \quad (3.6)
\]

\[
\frac{dC_2}{dt} = 1 - \delta_2 C_2(t, T), \quad (3.7)
\]

\[
\frac{dD}{dt} = -\frac{1}{2}(\rho_1^2 C_1^2(t, T) + \rho_2^2 C_2^2(t, T)), \quad (3.8)
\]

with the boundary conditions: $C_1(T, T) = 0$, $C_2(T, T) = 0$ and $D(T, T) = 0$. Solving for $C_1(t, T)$, $C_2(t, T)$, $D(t, T)$ yields

\[
C_1(t, T) = \frac{1 - e^{-\delta_1(T-t)}}{\delta_1}, \quad (3.9)
\]

\[
C_2(t, T) = \frac{1 - e^{-\delta_2(T-t)}}{\delta_2}, \quad (3.10)
\]

\[
D(t, T) = \frac{1}{2} \sum_{j=1}^{2} \frac{\rho_j^2}{\delta_j^3} \left[\frac{1}{2} (1 - e^{-2\delta_j(T-t)}) - 2(1 - e^{-\delta_j(T-t)}) + \delta_j (T-t)\right]. \quad (3.11)
\]

Hence, once the parameters ($\delta_1$, $\delta_2$, $\rho_1$ and $\rho_2$) are known, the forward survival curve can be computed analytically. This model is calibrated to age-cohorts of the population with data sourced from the Human Mortality Database, which spans from 1965 - 2011. We choose to model Australian males aged 50 for our purposes in discussing GMBs as an alternative Australian superannuation product. Since the GMBs are being marketed as investment products for superannuitants, the potential investors will be purchasing this product during their pre-retirement phase. Assuming that the policyholder retires at 65, the time to maturity is 15 years, well-within the long-term investment horizon. This set of specifications can be summarised as:

- $t = 0$ represents the calendar year 2011
- The individual’s age is $50 + t$ at time $t$ ($50$ at time $0$)

Figure 3.1 displays the force of mortality for the 50 year old cohort across various calendar years 1965 - 2011. The downward shift of $\mu(t)$ curves over the years illustrates the improvement in longevity. We have used the procedure outlined in Blackburn and Sherris (2013) to obtain the parameters (Table 3.1) for our chosen model. The calibration is based on the Kalman filter and we refer the interested reader to Blackburn and Sherris (2013) for further details on the estimation methodology. Furthermore, the estimated survival curve is plotted in Figure 3.2 along with its 90% confidence band. We assume that the insurer has up-to-date mortality data at his or her disposal. Since the Australian data in the Human Mortality Database only reaches up to 2011, we assume that the insurer is currently acting in the year 2011.

\footnote{The procedure is coded in MATLAB and is provided by Blackburn and Sherris (2013)}
Figure 3.1: Force of mortality for individuals aged 50 from 1965 – 2011

Figure 3.2: Projected Survival Function for 2011

From Figure 3.2, it is easy to see that there is an increasingly large confidence band in the survival function as we move further along into the future. The variability in future survival rates significantly affects the future profit and loss distributions of insurers, i.e. when the GMBs they have sold mature. This motivates insurers to participate in hedges which can reduce the
mortality risk in the future.

Table 3.1: Mortality model parameters

<table>
<thead>
<tr>
<th>$\delta_1$</th>
<th>$\delta_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1004</td>
<td>-0.1347</td>
<td>$1.4285 \times 10^{-4}$</td>
<td>$4.9659 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

3.2 Regime-Switching Models

In this section we utilise a regime-switching model to describe the financial market. We first define the model under the real-world probability measure, $\mathbb{P}$. Under the regime-switching framework, the market is incomplete and in order to make it complete, we need to incorporate the regime risk premiums and find an equivalent martingale measure. This can be done by using Esscher transforms (Gerber and Shiu, 1994). After applying these transforms, we are able to describe the fund dynamics under the risk-neutral measure $\mathbb{Q}$ and then continue with the pricing of GMBs. We adopt the procedure described in Shen et al. (2014) to demonstrate the change of measure under regime-switching models. We acknowledge that outside of the actuarial literature, there are alternative methods to the Esscher transform. A popular alternative is the minimal martingale measure (Follmer and Schweizer, 1991) whose aim is to minimise the distortion in the probability space, when moving from $\mathbb{P}$ to $\mathbb{Q}$.

Using the same filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we describe the state of the financial market by a continuous time, finite state Markov chain $X := \{X(t)| t \in T\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with a finite state space $S := \{s_1, s_2, ..., s_N\}$. The states can be interpreted as the “regimes” the economy undergoes where $N$ is the number of possible regimes. In the case of a two-regime model, the cardinality of $S$ is 2, i.e. $N = 2$. Elliott et al. (1994) note that, without loss of generality, we can represent the state space with a set of standard basis vectors, $B := \{e_1, e_2, ..., e_N\} \in \mathbb{R}^N$, where $e_j$ denotes a vector with 1 in the $j$-th coordinate and 0’s elsewhere.

Let $A$ be the $N \times N$ rate matrix for the chain $X$ under $\mathbb{P}$, with elements $[A]_{ij} = a_{ij}$, the constant transition intensity of the chain $X$ from regime $i$ to $j$. Elliott et al. (1994) obtain the following semi-martingale dynamics for the chain

$$X(t) = X(0) + \int_0^t AX(s)ds + M(t), \quad t \in T,$$

where $M(t)$ is a ($\mathcal{F}^X, \mathbb{P}$)-martingale and $\mathcal{F}^X$ is the natural filtration generated by the chain $X$. 

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3.2.1 Fund dynamics

We consider a continuous-time financial market consisting of two assets, a cash account and a risky fund whose value at time $t$ is $B(t)$ and $F(t)$, respectively. The time-$t$ risk-free force of interest, $r(t)$, time-$t$ drift rate of the fund, $\kappa(t)$, and the time-$t$ volatility of the fund, $\sigma(t)$, are all dictated by $X(t)$ such that

$$r(t) := \langle r, X(t) \rangle,$$  \hspace{1cm} (3.13)
$$\kappa(t) := \langle \kappa, X(t) \rangle,$$ \hspace{1cm} (3.14)
$$\sigma(t) := \langle \sigma, X(t) \rangle,$$ \hspace{1cm} (3.15)

where $\langle \cdot, \cdot \rangle$ denotes the inner product; $r, \kappa, \sigma$ are the vectors $(r_1, r_2, ..., r_N)'$, $(\kappa_1, \kappa_2, ..., \kappa_N)'$, $(\sigma_1, \sigma_2, ..., \sigma_N)'$ respectively, with $\kappa_j > r_j > 0$ and $\sigma_j > 0$ for each $j = 1, 2, ..., N$. We interpret the constants $r_j$, $\kappa_j$ and $\sigma_j$ as the risk-free force of interest, the fund’s mean drift rate and the fund’s volatility respectively under the regime $i$. Thus, for any time $t$, if $X(t) = e_j$, then $r(t) = r_j$, $\kappa(t) = \kappa_j$ and $\sigma(t) = \sigma_j$. The dynamics of the cash account, $B := \{B(t)|t \in T\}$, can be represented as

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1.$$ \hspace{1cm} (3.16)

Let $Z := \{Z(t)|t \in T\}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and is independent of $X$. The price process of the risky fund $F := \{F(t)|t \in T\}$ evolves over time according to the following stochastic differential equation:

$$\frac{dF(t)}{F(t)} = \kappa(t)dt + \sigma(t)dZ(t), \quad F(0) > 0.$$ \hspace{1cm} (3.17)

Equation (3.17) is very similar to that of the geometric Brownian motion, except that the parameters are not constant over time. If the number of regimes is one, the parameters are constant for all $t \in T$, and Equation (3.17) reduces to the GBM case. Let $Y(t) = \log \frac{F(t)}{F(0)}$ be the logarithm of the total return of the fund from time 0 to $t$. By Itô’s lemma, we have

$$dY(t) = \frac{dF(t)}{F(t)} - \frac{1}{2} \left( \frac{dF(t)}{F(t)} \right)^2 = \kappa(t)dt + \sigma(t)dZ(t) - \frac{\sigma^2(t)}{2}dt = \left( \kappa(t) - \frac{1}{2} \sigma^2(t) \right)dt + \sigma(t)dZ(t),$$ \hspace{1cm} (3.18)

which provides the real world dynamics of $Y(t)$ under $\mathbb{P}$. However, for pricing purposes, the dynamics of $Y(t)$ need to be under an equivalent risk-neutral measure $\mathbb{Q}$, and to achieve this, we apply the Esscher transform.
3.2.2 Esscher Transform

Elliott et al. (2005) show that the Esscher transform can be used to obtain a risk-neutral measure in the regime-switching models. We note that the Esscher transform is equivalent to the Girsanov (1960) theorem for finding the appropriate probability measure transformation, but in a regime-switching environment. We proceed to derive the dynamics for $Y(t)$ under a risk-neutral measure using the Esscher transform. Let $\theta(t)$ be the Esscher transform parameter which provides the local martingale condition and allows us to express the fund dynamics under $Q$. We denote the risk-neutral measure that is specified by $\theta(t)$ as $Q^{\theta}$. Similar to $r(t), \kappa(t), \sigma(t)$, $\theta(t)$ is also defined by the inner product between the vector $\theta$ and the Markov chain $X(t)$

$$
\theta(t) := \langle \theta, X(t) \rangle, \quad (3.19)
$$

where $\theta := (\theta_1, \theta_2, ..., \theta_N)'$ and $\theta_i$ is the Esscher transform parameter corresponding to the regime $i$. For all $t \in T$, we define the filtration $\mathcal{G} := \{\mathcal{G}(t) | t \in T \}$ where $\mathcal{G}(t) = \mathcal{F}^X(t) \vee \mathcal{F}^Y(t)$ and $\mathcal{F}^X(t), \mathcal{F}^Y(t)$ represent the filtrations generated by the processes $X(t)$ and $Y(t)$, respectively.

We define a $\mathcal{G}$-adapted exponential process $D^{\theta} := \{D^{\theta}(t) | t \in T \}$ by

$$
D^{\theta}(t) := \exp \left( \int_0^t \theta(s)dY(s) \right), \quad D^{\theta}(0) = 1. \quad (3.20)
$$

By Itô’s lemma, we have

$$
dD^{\theta}(t) = D^{\theta}(t)\theta(t)dY(t) + \frac{1}{2} D^{\theta}(t)\theta^2(t)dY^2(t)
= D^{\theta}(t) \left[ \theta(t) \left( \kappa(t) - \frac{1}{2} \sigma^2(t) \right) dt + \theta(t)\sigma(t)dZ(t) \right] + \frac{1}{2} \theta^2(t)dH^{\theta}(t),
\quad (3.21)
$$

where

$$
dH^{\theta}(t) = \theta(t) \left( \kappa(t) - \frac{1}{2} \sigma^2(t) \right) dt + \frac{1}{2} \theta^2(t)\sigma^2(t)dt + \theta(t)\sigma(t)dZ(t). \quad (3.22)
$$

Thus $H^{\theta} := \{H^{\theta}(t) | t \in T \}$ can be written in the following integral form:

$$
H^{\theta}(t) := \int_0^t \theta(s) \left( \kappa(s) - \frac{1}{2} \sigma^2(s) \right) ds + \frac{1}{2} \int_0^t \theta^2(s)\sigma^2(s)ds + \int_0^t \theta(s)\sigma(s)dZ(s). \quad (3.23)
$$

Now, the stochastic exponential can be used to solve Equation (3.21). The general solution to stochastic differential equations of the form

$$
dY(t) = Y(t)dX(t), \quad Y(0) = 1, \quad (3.24)
$$

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is the stochastic exponential of \( X(t) \), and is given by

\[
\mathcal{Y}(t) := \exp \left( X(t) - \frac{1}{2} \langle X \rangle(t) \right),
\]

where \( \langle X \rangle \) denotes the quadratic variation of process \( X \). Therefore, applying this result to Equation (3.23), we obtain

\[
\exp \left( \int_0^t \theta(s) dY(s) \right) = D^\theta(t) = \exp \left( H^\theta(t) - \frac{1}{2} \langle H^\theta \rangle(t) \right),
\]

\[
= \exp \left( \int_0^t \theta(s) \left( \kappa(s) - \frac{1}{2} \sigma^2(s) \right) ds + \int_0^t \theta(s) \sigma(s) dZ(s) \right). \tag{3.26}
\]

Our objective is to obtain a process that represents a \((\mathbb{G}, \mathbb{P})\)-martingale so that we can apply Girsanov’s theorem (Girsanov, 1960) to define the probability measure \( \mathbb{Q}^\theta \). Thus, we construct the \((\mathbb{G}, \mathbb{P})\)-martingale in \( \Lambda^\theta \). Let \( \Lambda^\theta := \{\Lambda^\theta(t) | t \in \mathcal{T}\} \) be a \( \mathbb{G} \)-adapted process as follows

\[
\Lambda^\theta(t) = \exp \left( \int_0^t \theta(s) dY(s) - J^\theta(t) \right), \tag{3.27}
\]

where \( J^\theta(t) \) denotes the predictable part of \( H^\theta(t) \). From Equation (3.23) it follows that

\[
J^\theta(t) = \int_0^t \left[ \theta(s) \left( \kappa(s) - \frac{1}{2} \sigma^2(s) \right) + \frac{1}{2} \theta^2(s) \sigma^2(s) \right] ds, \tag{3.28}
\]

and from Equations (3.26) and (3.28) we obtain

\[
\Lambda^\theta(t) = \exp \left( \int_0^t \theta(s) \sigma(s) dZ(s) - \frac{1}{2} \int_0^t \theta^2(s) \sigma^2(s) ds \right). \tag{3.29}
\]

By Itô’s lemma, we obtain

\[
d\Lambda^\theta(t) = \Lambda^\theta(t) \left( \theta(t) \sigma(t) dZ(t) - \frac{1}{2} \theta^2(t) \sigma^2(t) dt + \frac{1}{2} \theta^2(t) \sigma^2(t) dt \right)
= \Lambda^\theta(t) \theta(t) \sigma(t) dZ(t), \tag{3.30}
\]

and since \( \theta(t) \sigma(t) \) is bounded for \( t \in \mathcal{T} \), \( \Lambda^\theta \) is a \((\mathbb{G}, \mathbb{P})\)-local martingale. Consequently, we can define a new probability measure \( \mathbb{Q}^\theta \) equivalent to \( \mathbb{P} \) on \( \mathcal{G}(T) \) as follows:

\[
\left. \frac{d\mathbb{Q}^\theta}{d\mathbb{P}} \right|_{\mathcal{G}(T)} := \Lambda^\theta(T) \tag{3.31}
= \exp \left( \int_0^T \theta(s) \sigma(s) dZ(s) - \frac{1}{2} \int_0^T \theta^2(s) \sigma^2(s) ds \right). \tag{3.32}
\]
Next, we want to derive the local-martingale conditions. Traditionally, in the financial literature, the discounted price of the risky asset is set to be a martingale under the risk-neutral measure $Q^\theta$. By imposing the martingale condition, we assume that the market exhibits weak-form efficiency, i.e. all past prices are reflected in the current price, and that past prices are not informative of the future prices. The rationale behind this assumption is that there should not be any predictability in the stock returns after discounting by the risk-free force of interest plus the equity risk premium. Let us define the discounted price of the risky fund as follows:

$$F(t) := F(t) \exp \left( - \int_0^t r(s) ds \right).$$

(3.33)

For risk-neutral pricing, we require the discounted price, $\tilde{F}$, to be a $(\mathcal{G}, Q^\theta)$-local martingale. Elliott and Kopp (2004) show that if $\Lambda^\theta(t)\tilde{F}(t)$ is a $(\mathcal{G}, Q^\theta)$-local martingale, then $\tilde{F}$ is also a $(\mathcal{G}, Q^\theta)$-local martingale. Therefore, we can obtain our local-martingale conditions from observing $d(\Lambda^\theta(t)\tilde{F}(t))$. By applying the product rule for differentiation, we obtain

$$d\tilde{F}(t) = \exp \left( - \int_0^t r(s) ds \right) dF(t) - r(t) F(t) \exp \left( - \int_0^t r(s) ds \right) dt$$
$$= \tilde{F}(t) (\kappa(t) dt + \sigma(t) dZ(t)) - \tilde{F}(t) r(t) dt$$
$$= \tilde{F}(t) (\kappa(t) - r(t)) dt + \tilde{F}(t) \sigma(t) dZ(t),$$

(3.34)

and

$$d(\Lambda^\theta(t)\tilde{F}(t)) = \tilde{F}(t) d\Lambda^\theta(t) + \Lambda^\theta(t) d\tilde{F}(t) + d\tilde{F}(t) d\Lambda^\theta(t)$$
$$= \tilde{F}(t) \Lambda^\theta(t) \theta(t) \sigma(t) dZ(t) + \tilde{F}(t) \Lambda^\theta(t) \sigma(t) dZ(t)$$
$$+ \tilde{F}(t) \Lambda^\theta(t)(\kappa(t) - r(t)) dt + \tilde{F}(t) \Lambda^\theta(t) \sigma^2(t) \theta(t) dt.$$  

(3.35)

Now $\Lambda^\theta(t)\tilde{F}(t)$ is a $(\mathcal{G}, Q^\theta)$-local martingale if and only if the predictable part of the finite variation is indistinguishable from the zero process. That is

$$\tilde{F}(t) \Lambda^\theta(t)(\kappa(t) - r(t)) + \tilde{F}(t) \Lambda^\theta(t) \sigma^2(t) \theta(t) = 0.$$  

(3.36)

Thus, the local-martingale condition for $\tilde{F}$, which is equivalent to that of $\Lambda^\theta(t)\tilde{F}(t)$, is,

$$\kappa(t) - r(t) + \sigma^2(t) \theta(t) = 0, \quad \forall t \in \mathcal{T}.$$  

(3.37)

Equation (3.37) needs to hold for all times $t$ in $[0, T]$, which is equivalent to the requirement that it needs to hold for every regime. Thus, we obtain

$$\kappa_j - r_j + \sigma^2_j \theta_j = 0$$

(3.38)
for \( j = 1, 2, ..., N \). Using Girsanov’s theorem, we find the corresponding standard Brownian motion under \( Q^\theta \) as

\[
Z^\theta(t) = Z(t) - \int_0^t \frac{r(s) - \kappa(s)}{\sigma(s)} ds
\]

\[
= Z(t) - \int_0^t \theta(s) \sigma(s) ds,
\]

(3.39)

or

\[
dZ^\theta(t) = dZ(t) - \theta(t) \sigma(t) dt.
\]

(3.40)

Thus, under \( Q^\theta \), the risk-neutral measure, the dynamics of \( Y(t) \) are given by

\[
dY(t) = \left( \kappa(t) - \frac{1}{2} \sigma^2(t) \right) dt + \sigma(t) dZ(t)
\]

\[
= \left( \kappa(t) - \frac{1}{2} \sigma^2(t) \right) dt + \sigma(t) \left( dZ^\theta(t) + \theta(t) \sigma(t) dt \right)
\]

\[
= \left( \kappa(t) - \frac{1}{2} \sigma^2(t) + \sigma^2(t) \theta(t) \right) dt + \sigma(t) dZ^\theta(t)
\]

\[
= \left( r(t) - \frac{1}{2} \sigma^2(t) \right) dt + \sigma(t) dZ^\theta(t),
\]

(3.41)

and since \( F(t) = F(0) e^{Y(t)} \), we have

\[
dF(t) = r(t) F(t) dt + \sigma(t) F(t) dZ^\theta(t).
\]

(3.42)

Furthermore, the chain \( X \) has the following semi-martingale dynamics under \( Q^\theta \):

\[
X(t) = X(0) + \int_0^t A^\theta X(s) ds + M^\theta(t),
\]

(3.43)

where \( M^\theta(t) \) is a \((\mathcal{F}^X, Q^\theta)\) martingale. We set

\[
A^\theta = A,
\]

(3.44)

which is shown in Shen et al. (2014) to be equivalent to a single regime-switching model, i.e. the jump sizes in \( F \) are zero during a change in regimes.
3.3 Fourier Space Time-stepping

The purpose of the Fourier space time-stepping (FST) algorithm is to solve problems by first converting them into the Fourier space via Fourier transforms, and then converting them back to the real space. There are other various Fourier transform based methods which can be found in Carr and Madan, 1999, Hurd and Zhou, 2009, and Wong and Guan, 2009. All of these methods apply the Fast Fourier Transform (FFT), an algorithm which significantly reduces the computational cost of Fourier transforms, to provide an efficient option pricing framework. In this research, we use the FST algorithm described in Jackson et al. (2008).

3.3.1 Definitions and Properties

Let the continuous Fourier transform $\mathcal{G}[](k)$ of a function $g(y,t)$ be

$$\mathcal{G}[g(y,t)](k) = \int_{-\infty}^{\infty} g(y,t) e^{-2\pi iky} dy,$$

where

$$i = \sqrt{-1}. \quad (3.46)$$

This can be rewritten using $u = \frac{k}{2\pi}$ as

$$\mathcal{G}[g(y,t)](u) = \int_{-\infty}^{\infty} g(y,t) e^{-iuy} dy. \quad (3.47)$$

Let us denote the Fourier transform of the function $g(y,t)$ as $\hat{g}(u,t)$. Provided $\hat{g}(u,t)$, the original function can be recovered by the inverse Fourier transform. This is accomplished by the following transformation

$$\mathcal{G}^{-1}[\hat{g}(u,t)](y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(u,t) e^{iuy} du. \quad (3.48)$$

Furthermore, the Fourier transform of the $n$-th derivative with respect to $y$ is given by

$$\mathcal{G} \left[ \frac{\partial^n}{\partial y^n} g(y,t) \right] (u) = i^n u^n \int_{-\infty}^{\infty} g(y,t) e^{-iuy} dy.$$  

$$= (iu)^n \int_{-\infty}^{\infty} g(y,t) e^{-iuy} dy. \quad (3.49)$$
Often a discretised version of the Fourier transform is needed in numerical evaluations. To do this, the function \( g(y, t) \) is first evaluated at \( N \) distinct points, \( y_n \), with \( y_0 < y_1 < \ldots < y_{N-1} \). The vector of values \( g(y_n, t) \) is used as an approximation to the function \( g(y, t) \). The vector \( g(y_n, t) \) is then transformed into another equivalent vector in the Fourier space by the discrete Fourier transform (DFT):

\[
DFT[g(y, t)]_m = \sum_{n=0}^{N-1} g(y_n, t)e^{-2\pi i n (m-1) / N}, \quad m = 1, \ldots, N,
\]

where \( DFT[g(y, t)]_m \) denotes the \( m \)-th entry of the transformed vector. Similarly, the inverse discrete Fourier transform (IDFT) can be represented as

\[
IDFT[\hat{g}(u, t)]_m = \frac{1}{N} \sum_{n=0}^{N-1} g(u_n, t)e^{2\pi i n (m-1) / N}, \quad m = 1, \ldots, N.
\]

If the DFT is implemented exactly as specified in Equation (3.50), then the number of computations is of order \( N^2 \). The FFT solves Equations (3.50) and (3.51) with high accuracy and a lower computational order. We utilise the FFTW library in MATLAB to numerically implement the FFT algorithms. One of the main FFT algorithms used is the radix-2 Cooley-Tukey algorithm, which splits the DFT of a vector size \( N \) into two DFTs of size \( \frac{N}{2} \) each, on every recursion. For conciseness, we only outline one recursion for the DFT here, since the IDFT is the same as the DFT but has a positive sign in the exponent and an additional \( \frac{1}{N} \) factor. The summation used for the DFT in Equation (3.50) is split into even and odd indexes across \( n \):

\[
DFT[g(y, t)]_m = \sum_{n=0}^{\frac{N}{2}-1} g(y_{2n}, t)e^{-2\pi i \frac{2n(m-1)}{N}} + \sum_{n=0}^{\frac{N}{2}-1} g(y_{2n+1}, t)e^{-i2\pi \frac{(2n+1)(m-1)}{N}}
\]

\[
= \sum_{n=0}^{\frac{N}{2}-1} g(y_{2n}, t)e^{-2\pi i \frac{n(m-1)}{N/2}} + e^{-2\pi i \frac{m-1}{N}} \sum_{n=0}^{\frac{N}{2}-1} g(y_{2n+1}, t)e^{-2\pi i \frac{n(m-1)}{N/2}}
\]

\[
= E_m + e^{-i2\pi \frac{m-1}{N}} O_m,
\]

where \( E_m \) denotes the summation with even indexes of \( n \) and \( O_m \) denotes the summation with odd indexes of \( n \). Furthermore, due to periodicity of the Fourier transform,

\[
E_m = E_{m+N/2},
\]

\[
O_m = O_{m+N/2}.
\]
So, the each element of DFT vector of \( g(y,t) \) which is of length \( N \) can be evaluated through two DFT vectors, \( O \) and \( E \), both of length \( \frac{N}{2} \).

\[
DFT[g(y,t)]_m = \begin{cases} 
E_m + e^{-i2\pi \frac{m-1}{N}} O_m, & \text{for } 1 \leq m \leq \frac{N}{2}, \\
E_{m-N} + e^{-i2\pi \frac{m-1}{N}} O_{m-N}, & \text{for } \frac{N}{2} < m \leq N.
\end{cases}
\] (3.55)

Equations (3.52) - (3.55) describes one iteration of the radix-2 Cooley-Tukey algorithm. After applying this FFT algorithm, the number of computational steps required significantly decreases and becomes of order \( N \log_2 N \). The computational efficiency displayed by the FFT algorithm motivates its usage in option pricing.

Furthermore, after applying the Fourier transform to the infinitesimal generator of a Lévy process, we obtain its characteristic exponent:

\[
\mathcal{G}[\mathcal{L}g(y,t)](u) = \Psi(u)\mathcal{G}[g(y,t)](u),
\] (3.56)

where \( \mathcal{L} \) represents the infinitesimal generator of the process \( Y(t) \) and \( \Psi \) denotes the exponent of the characteristic function of \( Y(t) \). This is a useful property for solving Partial Integro-Differential Equations (PIDEs) which are used to describe the option pricing function. For our purposes of pricing GMBs, we write the price of the GMB as a function of the discounted log-return, \( \tilde{Y} \). In Appendix A, we prove that the discounted characteristic function, \( \phi \), is given by

\[
\phi(0,T,u) = \langle X(0) \exp \left[ \left( A + \text{diag}(g(u)) \right) T \right], 1 \rangle,
\] (3.57)

where \( A \) is the rate matrix that governs the transitions between regimes; \( 1 \) is a \( N \times 1 \) vector of ones; \( \text{diag}(g(u)) \) is a \( N \times N \) diagonal matrix with elements \( g_j(u) = -r_j + iu(r_j - \frac{1}{2}\sigma_j^2) - \frac{1}{2}u^2\sigma_j^2 \) for \( j = 1, 2, ..., N \) along its diagonal. Note that \( g_j(u) \) is the characteristic exponent for the GBM which is described in each regime and its corresponding coefficients. Thus, the characteristic exponent, as described in Equation (3.56), is

\[
\Psi(u) = \left( A + \text{diag}(g(u)) \right).
\] (3.58)

In the case of a two-regime model, i.e. \( N = 2 \), \( A = \begin{pmatrix} -a_{12} & a_{12} \\ a_{21} & -a_{21} \end{pmatrix} \), \( a_{12} > 0 \) and \( a_{21} > 0 \), we obtain

\[
\Psi(u) = \begin{pmatrix} -a_{12} + g_1(u) & a_{12} \\ a_{21} & -a_{21} + g_2(u) \end{pmatrix}.
\] (3.59)

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The discounted characteristic function \( \phi \), can also be used to derive the probability density function of the discounted log-return, \( \tilde{Y} \), with the use of Fourier transforms. The characteristic function is defined as

\[
\phi(u) = \int_{-\infty}^{\infty} f(\tilde{y})e^{iu\tilde{y}}d\tilde{y} = 2\pi G^{-1}[f(\tilde{y})](u),
\]

where \( f(\tilde{y}) \) is the density function. After applying the results from Equations (3.47) and (3.48) to the density, we obtain

\[
f(\tilde{y}) = \frac{1}{2\pi} G[\phi(u)](\tilde{y}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(u)e^{-iu\tilde{y}}du.
\]

Figure 3.3 plots the density functions in each regime under the risk-neutral measure (since it is risk-neutral, the drift rate of the fund does not affect the process of \( \tilde{Y} \) and thus we omit it). We construct Regime 1 to describe a “bull” market environment, where as the Regime 2 describes a “bear” market. Thus, Regime 2 is characterised by higher volatility and lower force of interest. This is observed in the figure as Regime 2 has a wider spread of returns than Regime 1. Furthermore, in comparison to Regime 2, the higher force of interest in Regime 1 induces a higher mean return.
3.3.2 Application of the FST algorithm in a regime-switching environment

Traditionally, for option pricing, one needs to solve the option price function that is governed by its relevant Partial Differential Equation (PDE) or Partial Integro-Differential Equation (PIDE). The FST method outlined in Jackson et al. (2008), provides option pricing based on the solution in the Fourier domain. This differs to usual finite difference schemes used in option pricing, which solve the PIDE in the real domain. Jackson et al. (2008) demonstrate that the FST method is robust and can be easily adapted for path dependence in various options. Furthermore, during the numerical implementation, the FST method does not require the analytic expression for the Fourier transform of the terminal payoff, and thus it can handle more exotic forms of the payoff functions. Another advantage of the FST method is that its computational time is proportional to the number of regimes.

Here, we outline the procedure for pricing options under the regime-switching Log-Normal (RSLN) model. Let \( v(Y(t), t, X(t)) \) denote the price function of an option at time \( t \), which is dependent on the time-\( t \) log-return \( Y(t) \) and the time-\( t \) regime \( X(t) \). Furthermore, we can write the function \( v \) as an inner product

\[
v(Y(t), t, X(t)) = \langle v(Y(t), t), X(t) \rangle,
\]

where \( v(Y(t), t) \) denotes the vector of values for \( v(Y(t), t, X(t)) \) across all \( N \) states of \( X(t) \), i.e. \( v(Y(t), t) = (v(Y(t), t, e_1), v(Y(t), t, e_2), ..., v(Y(t), t, e_N))' \). Applying Itô’s lemma to \( v \) under \( Q \), we obtain

\[
dv = \left( \frac{\partial}{\partial t} + (r(t) - \frac{1}{2}\sigma^2(t)) \frac{\partial}{\partial Y} + \frac{1}{2}\sigma^2(t) \frac{\partial^2}{\partial Y^2} \right) v(Y(t), t, X(t)) + \langle v(Y(t), t), AX(t) \rangle.
\]

Furthermore, under the risk-neutral measure, \( dv = r(t)v \). So we have

\[
\left( \frac{\partial}{\partial t} + (r(t) - \frac{1}{2}\sigma^2(t)) \frac{\partial}{\partial Y} + \frac{1}{2}\sigma^2(t) \frac{\partial^2}{\partial Y^2} - r(t) \right) v(Y(t), t, X(t)) + \langle v(Y(t), t), AX(t) \rangle = 0,
\]

(3.64)

\[
\left( \frac{\partial}{\partial t} + \mathcal{L} \right) v(Y(t), t, X(t)) + \langle v(Y(t), t), AX(t) \rangle = 0,
\]

(3.65)

where \( \mathcal{L} \) is the infinitesimal generator for the discounted process of \( Y(t) \). For notational convenience, let \( v^{(k)}(y, t) = v(Y(t) = y, t, X(t) = e_k) \). After expanding Equation (3.65) for all regimes
1 to $N$, we have the option price function satisfying the following system of PDEs for all $k$:

\begin{align}
(\partial_t + (a_{kk} + L^{(k)}) v^{(k)}(y,t)) + \sum_{j \neq k} a_{kj} v^{(j)}(y,t) &= 0, \\
 v^{(k)}(y,T) &= H,
\end{align}

(3.66)

(3.67)

where $L^{(k)}$ is the infinitesimal generator under regime $k$; $a_{ij}$ are the $(i,j)$-th entry in the rate matrix $A$; $H$ is the terminal condition occurring on the maturity date $T$. For example, in the case of a European call with strike price $K$, we have $H = \max(F(0)e^y - K, 0)$. Next, apply Fourier transforms to Equations (3.66) and (3.67) and use the property in Equation (3.56) to obtain

\begin{align}
(\partial_t + a_{kk} + \Psi^{(k)}(u)) G[v^{(k)}(y,t)](u) + \sum_{j \neq k} a_{kj} G[v^{(j)}(y,t)](u) &= 0, \\
G[v^{(k)}(y,T)](u) &= G[H](u),
\end{align}

(3.68)

(3.69)

where $\Psi^{(k)}(u)$ denotes the characteristic exponent under each regime. In the RSLN model, the fund evolves under the GBM under each regime as defined in Equation (3.42), so that the characteristic exponent under the risk-neutral measure becomes

$$
\Psi^{(k)}(u) = i \left( r_k - \frac{\sigma_k^2}{2} \right) u - \frac{\sigma_k^2 u^2}{2} - r_k = g_k(u),
$$

(3.70)

where $r_k$ and $\sigma_k$ are the force of interest and volatility under the regime $k$, respectively. Equations (3.68) and (3.69) can be rewritten in matrix form:

\begin{align}
(\partial_t + \Psi(u))G[v(y,t)](u) &= 0, \\
G[v(y,T)](u) &= G[H](u)1,
\end{align}

(3.71)

(3.72)

where 1 is a $N \times 1$ vector of ones; $v$ is the vector of option price functions consisting of $v^{(k)}$; $\Psi(u)$ is the matrix characteristic function and its elements are

$$
\Psi(u)_{kl} := \begin{cases} 
A_{kk} + \Psi^{(k)}(u) & \text{if } k = l, \\
A_{kl} & \text{if } k \neq l.
\end{cases}
$$

(3.73)

The result here matches those previously presented in Equation (3.58) and we can also write $\Psi(u) = A + diag(g(u))$. If there is no embedded exotic feature like a knock-out or early exercises between $t$ and $T$, then Equation (3.71) can be integrated in a single time step to find $G[v(y,t)](u)$. For example, the European style options do not have any extra features between $t$ and $T$, and

$$
G[v(y,t)](u) = \exp \left( (T - t) \Psi(u) \right) G[H](u)1.
$$

(3.74)
Taking the inverse transform provides the vector of values in the real domain. Thus, we obtain
\[ v(y, t) = G^{-1} \left[ e^{\Psi(u)(T-t)} \cdot G[v(y, T)](u) \right], \] (3.75)
with each entry signifying the value corresponding to the regime at time \( t \). In the numerical implementation, the continuous Fourier transform \( G(\cdot) \) is approximated by the DFT, which is then implemented through the FFT such as the Cooley-Tukey algorithm, briefly outlined in Equation (3.55). Thus, the numerical evaluation of \( v \) is formally represented as
\[ v(t) = FFT^{-1} \left[ e^{\Psi(u)(T-t)} \cdot FFT[v(T)](u) \right], \] (3.76)
where \( FFT \) and \( FFT^{-1} \) denote the application of the FFT algorithm to the DFT and IDFT in Equations (3.50) and (3.51), respectively. From Equation (3.76), it is straightforward to see that only the numerical computation of a DFT vector is needed and that the analytic expression for the Fourier transform of \( v(T) \) is not required. This makes the FST method particularly useful in evaluating more complex options such as the ones seen in Guaranteed Minimum Withdrawal Benefits.

It follows from Equation (3.76) that the price of the option at time 0 is given by
\[ p_0 = \pi_0' v(0), \] (3.77)
where \( \pi_0 \) represents the initial vector of probabilities of each regime. This vector is often estimated via the Kalman filter or by using a weighted average such that the price is equivalent to the market observed value. Since we are not interested in the calibration component, we can assume that \( \pi_0 \) is known.

Furthermore, the forward time-\( t \) value of the option can also be calculated as
\[ p_t = \pi_t' v(t), \] (3.78)
where \( \pi_t' \) is the probability of states at time \( t \) given the initial vector of probabilities \( \pi_0' \). The formula for \( \pi_t' \) is given by
\[ \pi_t = \pi_0' \cdot \left[ Pr(X_t = j|X_0 = i) \right]_{ij}, \]
\[ = \pi_0' \left( \frac{a_{21}}{a_{12} + a_{21}} + \frac{a_{12}}{a_{12} + a_{21}} e^{-(a_{21} + a_{12})t} + \frac{a_{12}}{a_{12} + a_{21}} e^{-(a_{21} + a_{12})t} \right). \] (3.79)

This completes the main framework of option pricing under the FST method which is graphically represented in Figure 3.4. However, as stated in Equation (3.74), we have ignored the case where
there are embedded exotic features. The FST method deals with these features by evaluating $v$ at each point along the time grid $t_j \in [0, T]$, where $0 = t_0 < \ldots < t_{j-1} < t_j < \ldots < T$, and the exotic features are implemented by adjusting the value at each time-step accordingly:

$$v(t_{j-1}^+) = \text{FFT}^{-1}[\text{FFT}(v(t_j)) \cdot e^{\Psi(t_j - t_{j-1})}],$$

$$v(t_{j-1}) = H(v(t_{j-1}^+)),$$

where $t_{j-1}^+$ denotes the time right after $t_j$ and $H$ is the adjustment function. A finer grid for $t_k$ results in a more accurate solution, albeit at the cost of higher computational steps. In the following, we provide an example of a path-dependent option and its relevant adjustment function $H$ - an American call option. The value of the American option is calculated with the assumption that the option is exercised at the optimal point in time, when the option has the largest value. Then, the adjustment function at each time point is

$$H(v(t_j)) = \max\left(v(t_{j-1}^+), v(t_j)\right).$$

From Equation (3.82), it is easy to see that by $t_0$, the value of the option is the maximum of the option values throughout the time grid. It follows that given any particular exotic feature in the option, one is able to find a suitable $H$ that allows the option to be priced.
3.4 Valuation of GMBs

In the GMMB, GMIB and GMDB, the contract’s payoff is path independent and can be described by a combination of European calls and zero-coupon bonds, whereas for a GMWB, the payoff is path dependent. Thus, in our pricing framework, we have separated the set of GMBs by their payoff behaviour. In this section we discuss the more elementary problem first - the path independent derivatives within the GMMBs, GMIBs and GMDBs. We denote the value of these various GMBs with a subscript representing their characteristic, i.e. GMMB, GMIB and GMDB as $V_M, V_I$ and $V_D$ respectively.

3.4.1 Valuation of GMMBs, GMIBs, GMDBs

1. Guaranteed Minimum Maturity Benefit (GMMB)

We begin our pricing methodology by considering the simplest case, the GMMB. Let $V_M(t, T, G)$ denote the time-$t$ value of a GMMB rider that matures at time $T$ with a guarantee of $G$. The GMMB guarantees the minimum level of benefit at maturity conditional on the policyholder survival, that is

$$V_M(t, T, G) = E^Q[1_{\{t_d > T - t\}}e^{-\int_t^T r(s)ds} \max(G, F(T))]$$

(3.83)

where $t_d$ is the time until death for a person aged $50 + \tau$ at time $t$; $r(s)$ is the risk-free force of interest at time $s$; and $F(T)$ is the fund value at maturity. Recall from Section 3.1 that $t = 0$ specifies the calendar year 2011 and that the policyholder’s age is represented as $50 + \tau$. We separate the mortality component from the financial component as independence is assumed between the mortality process and the financial market. Thus, the value of the GMMB becomes

$$V_M(t, T, G) = E^Q[1_{\{t_d \geq T - t\}}e^{-\int_t^T r(s)ds} \max(G, F(T))]$$

(3.84)

The values for $S(t, T)$ can be obtained from Equation (3.1). For the financial component, we decompose the benefit into a European call, $C$, written on the fund $F$, with strike $G$, maturity $T$ and a unit zero-coupon bond $B$ which also matures at time $T$. Denoting the time-$t$ value of the call as $C(F, t, T)$ and the time-$t$ value of the bond as $B(t, T)$, we obtain

$$E^Q[1_{\{t_d \geq T - t\}}e^{-\int_t^T r(s)ds} \max(G, F(T))] = GB(t, T) + C(F, t, T).$$

(3.85)
Thus, it holds that
\[
V_M(t, T, G) = S(t, T)(GB(t, T) + C(F, t, T)). \tag{3.86}
\]

Furthermore, we can show that the GMIB and GMDB are also comprised of the same three elementary factors, namely the survival factor \(S(t, T)\), the zero-coupon bond \(B(t, T)\) and the European call \(C(F, t, T)\).

2. Guaranteed Minimum Income Benefit (GMIB)

In the GMIB, the policyholder is guaranteed a minimum level of income, \(G\), as long he or she stays alive, until maturity \(T\). With the assumption that the income payments are paid annually, the value of the GMIB rider equals the summation of GMMBs with increasing maturities. Hence, denoting the GMIB rider as \(V_I(t, T, G)\), we obtain
\[
V_I(t, T, G) = \sum_{j=t+1}^{T} V_M(t, j, G) \tag{3.87}
\]
\[
= \sum_{j=t+1}^{T} S(t, j)(GB(t, j) + C(F, t, j)). \tag{3.88}
\]

3. Guaranteed Minimum Death Benefit (GMDB)

In the GMDB, the policyholder’s beneficiaries are paid a guaranteed a minimum level of benefit in the event of the policyholder dying before the maturity of the contract. The immediate assumption is that the benefit is paid immediately on death, and so the value of a GMDB rider in the continuous setting is
\[
\tilde{V}_D(t, T, G) = \mathbb{E}^Q[\mathbf{1}_{\{t \leq T-t\}} e^{-\int_{t}^{t+d} r(s)ds} \max(F(t + t_d), G)]
\]
\[
= \int_t^T \mathbb{E}^Q[\mu(z)S(t, z)e^{-\int_{t}^{z} r(s)ds} \max(F(z), G)] dz, \tag{3.89}
\]
where \(S(t, z)\) is the probability that a person aged 50 + \(t\) at time \(t\) is alive at time \(z\); \(\mu(z)\) is the instantaneous death rate for a person aged 50 + \(z\) at time \(z\); \(G\) is the minimum guaranteed benefit received on death; and \(F(z)\) is the fund value at time \(z\). Again, we separate the mortality component from the financial component and utilise Equation (3.85) to obtain
\[
\tilde{V}_D(t, T, G) = \int_t^T \mathbb{E}^Q[\mu(z)S(t, z)] \mathbb{E}^Q[e^{-\int_{t}^{z} r(s)ds} \max(F(z), G)] dz
\]
\[
= \int_t^T \mathbb{E}^Q[\mu(z)S(t, z)] \left(GB(t, z) + C(F, t, z)\right) dz. \tag{3.90}
\]
Evaluating Equation (3.90) numerically can be very time-consuming. Instead, we discretise
the time intervals annually, and assume that the payout of GMDBs is calculated at the end of every year, which is consistent with real-life contract specifications. Thus, rather than the insurer immediately providing the benefit on death, payments are deferred to end of the year. Hence \( \tilde{V}_D(t, T, G) \) from Equation (3.90) is annually discretised into

\[
V_D(t, T, G) = \sum_{j=t+1}^{T} \left[ S(t, j - 1) - S(t, j) \right] \left( GB(t, z) + C(F, t, z) \right).
\] (3.91)

It is easy to see that the values of the GMMB, GMIB and GMDB can be decomposed into three components: the survival factor, the zero-coupon bond, and the European call. For the mortality component, it is straightforward to evaluate \( S(t, T) \) using the Equations (3.5), (3.9), (3.10), (3.11) with the parameters provided in Table 3.1. This leaves us with only two unknown quantities to be evaluated, the European call and a zero-coupon bond.\(^4\)

Firstly, for the European call, we apply the FST method for path-independent options and from Equations (3.76) and (3.78), we obtain

\[
C(F, t, T) = \pi'_t G^{-1} \left[ e^{\Psi(u)(T-t)} \cdot G[\max(F(t)e^{Y(T)-Y(t)} - G, 0)](u)1 \right],
\] (3.92)

which is numerically evaluated as

\[
C(F, t, T) = \pi'_t FFT^{-1} \left[ e^{\Psi(u)(T-t)} \cdot FFT[\max(F(T) - G, 0)](u)1 \right].
\] (3.93)

Next, for the zero-coupon bond, we require some elementary results on the occupation time inside states for Markov chains. The two regimes which describe the economy are indicated through the chain \( X \), where \( X = (1, 0)' \) for regime 1 and \( X = (0, 1)' \) for regime 2. Let us define \( I(t) = 1 \) if \( X(t) = (1, 0)' \) and \( I(t) = 0 \) if \( X(t) = (0, 1)' \). Then the occupation time in regime 1, \( T_{t,T} \), for the time period \([t, T]\) is given by

\[
T_{t,T} = \int_{t}^{T} I(s) ds,
\] (3.94)

and it holds that the time-\( t \) value of the zero-coupon bond is given by

\[
B(t, T) = E^Q[e^{-\int_{t}^{T} r(s) ds}] = E^Q[e^{-\int_{t}^{T} (r_1 \tau_{t,T} + r_2 (T-t-\tau_{t,T}))}] = e^{-r_2(T-t)}E^Q[e^{-(r_1-r_2)\tau_{t,T}}].
\] (3.95)

\(^4\)One can also apply another change of numeraire to remove the discounting component, \( B(t, T)^{-1} \).
Darroch and Morris (1968) provide the moment generating function of $T_u, T_\tau$ as

$$\psi_{t,T}^{\tau}(u) = E^Q(e^{uT_u,T_\tau}) = \pi_t' e^{(T-t)(A^\theta + uD)}1,$$  

(3.96)

where $\pi_t$ is the time-$t$ vector of probabilities over states 1 and 2, $A^\theta = A = \begin{pmatrix} -a_{12} & a_{12} \\ a_{21} & -a_{21} \end{pmatrix}$ is the rate matrix of chain $X$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then,

$$B(t,T) = e^{-r_2(T-t)} \psi_{t,T}^{\tau}(-(r_1 - r_2))$$

$$= e^{-r_2(T-t)} \pi_t' e^{(T-t)(A-(r_1-r_2)D)}1$$

$$= e^{-r_2(T-t)} \pi_t' \exp\left(\begin{pmatrix} -(a_{12} - r_1 + r_2)(T-t) \\ a_{21}(T-t) & a_{12}(T-t) & -a_{21}(T-t) \end{pmatrix}\right)1.$$

(3.97)

Thus, every component for pricing GMMBs, GMIBs and GMDBs in Equations (3.86), (3.88) and (3.91) can be obtained numerically. This completes the risk-neutral pricing of GMMBs, GMIBs and GMDBs.

3.4.2 GMWB Valuation

In this section, we will provide the no-arbitrage pricing framework for the Guaranteed Minimum Withdrawal Benefit (GMWB). First, let us formally describe the GMWB contract. At inception, the policyholder pays a lump sum to the insurer, which becomes the initial balance of two accounts, the sub-account and the guarantee account. For simplicity, we assume that the GMWB only allows withdrawals on a regular annual basis. When the policyholder withdraws an amount, $\gamma$, the guarantee account and the sub-account value decreases by $\gamma$ as well. The policyholder is able to withdraw as long as the guarantee account is above zero, no matter how the sub-account is performing. On maturity, the policyholder receives the larger of the sub-account balance or the guarantee account balance, less any fees. In the existing literature (Milevsky and Salisbury, 2006, Dai et al., 2008), there are two predominant styles of GMWBs - when the policyholder is only allowed to withdraw a pre-determined amount, otherwise known as the static withdrawal case; or when the policyholder withdraws the optimal amount at each withdrawal date, which is referred to as the dynamic withdrawal case. In the following we will show how our pricing methodology and the FST algorithm can be applied to solve the dynamic withdrawal problem and static withdrawal problem.

To demonstrate the flexibility of the FST algorithm, we first describe the pricing problem under

5Withdrawals can be continuous (see Dai et al., 2008), or the time between withdrawal dates can be non-constant.
dynamic withdrawals and later show that the static withdrawal is a special case where the withdrawal amount is set to be constant. Let $W(t)$ denote the balance of the sub-account, and $A(t)$ be the balance of the guarantee account. Thus, at inception, $W(0) = A(0)$, and furthermore, the guarantee account decreases over time, i.e., $A(t) \in [0, W(0)]$. Let $U(W, A, t)$ denote the time-$t$ value of a GMWB rider, and $V(W, A, t)$ be the total value of the variable annuity with an embedded GMWB rider at time $t$. Thus $V$ satisfies the following relationship$^6$

$$V = U + W. \quad (3.98)$$

Since the price of the variable annuity $V$ at time 0 is fixed to be equal to the guarantee account, the insurer prices the GMWB through charging a proportional fee $\alpha$ on the returns of the sub-account $W$ which is invested into a risky fund $F$. Much of the pricing literature for GMWBs have focused on finding the fair fee (see Milevsky and Salisbury, 2006, Dai et al., 2008, Chen et al., 2008, Peng et al., 2012). Furthermore, Chen et al. (2008) argue that the fee charged can be split into a fee paid for the provision of the guarantee, $\alpha_g$, and a mutual fund management fee, $\alpha_m$, such that

$$\alpha_{tot} = \alpha_g + \alpha_m, \quad (3.99)$$

where $\alpha_{tot}$ denotes the total fee charged. The distinction is made so that only the guarantee fee $\alpha_g$ can be used to fund the GMWB guarantee. For the remainder of this section, we shall describe the GMWB with a management fee similar to that specified in Chen et al. (2008) and Lippa (2013), with the difference being that the underlying fund follows a RSLN model rather than a GBM model.

We now describe the dynamics of the guarantee account $A$, which depend only on the withdrawal amount and frequency. Let $t_k$ be the $k$-th withdrawal time for $k = 0, ..., T$. However, for computational convenience, it is simpler to formulate the problem with a time to maturity index, which we denote as $\tau$ and we have $\tau = T - t$. Thus, we convert the withdrawal times $t_k$ for $k = 0, ..., T$ into the $\tau_k$ for $k = 0, ..., T$, going backwards in time with $\tau_0 = 0$ and $\tau_T = T$. This change in variable is graphically represented in Figure 3.5. Let $\gamma_k$ be the control variable representing the withdrawal amount at time $\tau = \tau_k$. Furthermore, since the policyholder is only

$^6$ We suppress the dependence of $V$ on the variables $W, A, t$ for notational convenience.
allowed to withdraw an amount less than or equal to the guarantee account, then $\gamma_k \in [0, A]$. Thus, we have

$$dA = \begin{cases} -\gamma_k & \text{if } \tau = \tau_k, = \\ 0 & \text{otherwise.} \end{cases}$$

(3.100)

Next, we describe the dynamics of the sub-account, $W$. Although the sub-account $W$ is invested into the risky fund $F$, the fee imposed for the provision of the GMWB alters its returns, and thus the dynamics of $W$ is different to the dynamics of $F$ introduced in Equation (3.17). This is a key difference which makes the pricing of GMWBs different to other types of GMBs, as previously described in Section 3.4. The dynamics of $W(t)$ are given by

$$dW(t) = \begin{cases} \left(\kappa(t) - \alpha_{tot}\right)W(t)dt + \sigma(t)W(t)dZ(t) + dA(t) & \text{if } W(t) > 0, \\ 0 & \text{if } W(t) = 0, \end{cases}$$

(3.101)

where $\kappa(t)$ and $\sigma(t)$ are the drift rate and volatility of the fund under the regime-switching environment specified in Equations (3.14) and (3.15); $Z$ is a Brownian motion under the real-world probability measure. Figure 3.6 shows the relationship between a hypothetical sub-account and guarantee account.

![Figure 3.6: Value of the sub-account W and the guarantee account A](image)

Let $G_k$ be the guaranteed withdrawal rate at $\tau_k$ as stated in the contract of the GMWB. Often,
$G_k$ is set such that $G_k = \frac{W(0)}{T}$, for all $k$, which ensures that the guarantee account is depleted at maturity. In the case of static withdrawals, the policyholder must withdraw at the prescribed rate $G_k$, unless there is insufficient value left in the guarantee account. Furthermore, the cashflow received by the policyholder is equal to the amount withdrawn. The static withdrawal can be described as\footnote{The minimum function prevents overdrawing the current guarantee account value}

$$f(\gamma^s_k) = \gamma^s_k, \quad (3.102)$$

and

$$\gamma^s_k = \min(A, G_k), \quad (3.103)$$

where $\gamma^s_k$ denotes the withdrawal amount under the static withdrawal scheme; and $f(\gamma^s_k)$ is the amount received by the policyholder. However, in the dynamic withdrawal case, the policyholder is free to choose the amount to withdraw. It is common to see the insurer charging an extra surrender fee $\eta_k$ to disincentivise policyholders from withdrawing more than the guaranteed amount. Denoting $\gamma^d_k$ to be the dynamic withdrawal amount, we have a similar set of equations to Equations (3.102) and (3.103):

$$f(\gamma^d_k) = \begin{cases} 
\gamma^d_k & \text{if } 0 \leq \gamma^d_k \leq G^k, \\
\gamma^d_k - \eta_k(\gamma^d_k - G_k) & \text{if } \gamma^d_k > G_k, 
\end{cases} \quad (3.104)$$

and

$$\gamma^d_k \in [0, A], \quad (3.105)$$

where $f(\gamma^d_k)$ is the amount received by the policyholder under the dynamic withdrawal scheme. Furthermore, the surrender charge $\eta_k$ at each withdrawal time is typically time-dependent and decreases over time to zero. Table 3.2 shows a hypothetical specification for $\eta_k$.

From the illustration in Figure 3.6, we find that the terminal condition of $U$ is

$$U(W, A, t = T) = \max(A(1 - \eta) - W, 0), \quad (3.106)$$

i.e.

$$U(W, A, \tau = 0) = \max(A(1 - \eta) - W, 0). \quad (3.107)$$
Table 3.2: Time-dependent surrender charges $\eta_k$.

<table>
<thead>
<tr>
<th>Year</th>
<th>$\eta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq t &lt; 2$</td>
<td>8</td>
</tr>
<tr>
<td>$2 \leq t &lt; 3$</td>
<td>7</td>
</tr>
<tr>
<td>$3 \leq t &lt; 4$</td>
<td>6</td>
</tr>
<tr>
<td>$4 \leq t &lt; 5$</td>
<td>5</td>
</tr>
<tr>
<td>$5 \leq t &lt; 6$</td>
<td>4</td>
</tr>
<tr>
<td>$6 \leq t &lt; 7$</td>
<td>3</td>
</tr>
<tr>
<td>$t \geq 7$</td>
<td>0</td>
</tr>
</tbody>
</table>

Similarly, for $V$, we combine Equations (3.98) and (3.107) to obtain its terminal condition in the following way:

$$V(W, A, \tau = 0) = \max(A(1 - \eta), W),$$

(3.108)

which can be interpreted as the final payment to the policyholder on maturity. In Appendix B, we use no-arbitrage hedging arguments and Itô’s lemma, to show that between withdrawal dates, $V$ satisfies the following PDEs under each regime $p$:

$$\frac{\partial}{\partial \tau} V^{(p)} = (A_{pp} + \tilde{L}^{(p)}) V^{(p)} + \sum_{q \neq p} A_{pq} V^{(q)} + \alpha_m W,$$

(3.109)

where $\alpha_m W$ is the component describing the effect of the mutual fund management fees; $A_{pq}$ is the $(p, q)$-th element in the rate matrix $A$ of the Markov chain $X$; and $\tilde{L}^{(p)}$ is the $\alpha_{tot}$-adjusted operator such that

$$\tilde{L}^{(p)} = \frac{1}{2} \sigma_p^2 W^2 \frac{\partial^2}{\partial W^2} + (r_p - \alpha_{tot}) W \frac{\partial}{\partial W} - r_p.$$

(3.110)

Here, $r_p$ and $\sigma_p$ denote the risk-free force of interest and volatility under the $p$-th regime. Thus, we have described $V$ at maturity and in between withdrawal times, and so only the behaviour of $V$ on the withdrawal dates $\tau_k$ remains to be discussed. At $\tau_k$, $V$ satisfies the following conditions, depending on the type of withdrawal:

- Under static withdrawals$^8$

$$V(W, A, \tau_{k+}) = V(\max(W - \gamma_k, 0), A - \gamma_k, \tau_k) + f(\gamma_k), k = 0, \ldots, K - 1.$$  

(3.111)

$^8$ $\tau_{k+}$ refers to the moment before withdrawal in real-time, but we use the “+” notation since $\tau = T - t$ represents “backwards” time.
• Under dynamic withdrawals

\[
\mathcal{V}(W, A, \tau_k) = \sup_{\gamma^d_k \in [0, A]} [\mathcal{V} \left( \max(W - \gamma^d_k, 0), A - \gamma^d_k, \tau_k \right) + f(\gamma^d_k)], k = 0, ..., K - 1. \tag{3.112}
\]

Equation (3.112) also carries the interpretation that the policyholder selects a withdrawal amount that maximises the total value of the annuity after withdrawal. Next, given Equations (3.108)-(3.112), we utilise the FST algorithm in the same way as described in Section 3.3.2. After applying the FFT transforms to the PDEs and terminal conditions (Equations (3.108) and (3.109)), we obtain the following representation in the Fourier Space

\[
\frac{\partial}{\partial \tau} G[V^{(p)}](u) = (A_{pp} + \tilde{\Psi}^{(p)}(u)) G[V^{(p)}](u) + \sum_{q \neq p} A_{pq} G[V^{(q)}](u) + \alpha_m G[W](u), \tag{3.113}
\]

\[
G[V^{(p)}](u) = G[\max(A(1 - \eta), W)](u), \tag{3.114}
\]

and \(\tilde{\Psi}^{(p)}\) denotes the fee adjusted characteristic exponent under each regime

\[
\tilde{\Psi}^{(p)}(u) = i \left( r_p - \alpha_{tot} - \frac{\sigma^2_p}{2} \right) u - \frac{\sigma^2_p u^2}{2} - r_p. \tag{3.115}
\]

Equations (3.113) and (3.114) describe \(\mathcal{V}\) under each regime, \(p\), and can be collapsed into matrix form as follows:

\[
\left( - \frac{\partial}{\partial \tau} + \tilde{\Psi} \right) G[\mathbf{V}](u) + \alpha_m G[W](u) \mathbf{1} = 0, \tag{3.116}
\]

\[
G[\mathbf{V}(W, A, \tau = 0)](u) = G[\max(A(1 - \eta), W)](u) \mathbf{1}, \tag{3.117}
\]

where \(\mathbf{1}\) is a \(N \times 1\) vector of ones; \(\mathbf{V}\) is the vector of variable annuity price functions under each regime; \(\Psi_W(u)\) is the matrix characteristic function and its elements are

\[
\tilde{\Psi}(u)_{pq} := \begin{cases} 
A_{pp} + \tilde{\Psi}^{(p)}(u) & \text{if } q = p, \\
A_{pq} & \text{if } q \neq p.
\end{cases} \tag{3.118}
\]

Since the FST algorithm only finds the value through stepping in time, we adjust the value at each time-step accordingly as in Equations (3.111) and (3.112) to account for the path-
dependence created by the withdrawals. We obtain

\[ V(W, A, \tau_{k+1}) = FFT^{-1}\left[ FFT[V(W, A, \tau_k)] \cdot e^{\hat{\Phi}(\tau_{k+1} - \tau_k)} + FFT[\alpha_m W] \right], \tag{3.119} \]

where the appropriate form of Equation (3.120) is used depending on whether the problem is set as a static withdrawal or a dynamic withdrawal, respectively. It is also important to account for the wrap-around error which is induced when applying the FFT. We leave the discussion for the wrap-around error, and methods to reduce this error in Appendix C.

So far, the variable annuity with the GMWB rider has been priced under the assumption of no mortality. If the policyholder dies before the maturity date \( T \), then the variable annuity effectively matures and pays out the greater of the sub-account and guarantee account. For simplicity, and similar reasons outlined in the pricing of GMDBs in Section 3.4.1, we assume that this payment is made at the next withdrawal time. Suppose the policyholder dies within the time interval \([\tau_j, \tau_{j+1})\), then, the value of the variable annuity is simply adjusted so that the terminal condition is enforced at time \( \tau_j \), i.e.

\[ V(W, A, \tau = \tau_j) = \max(\mathcal{A}(1 - \eta), W), \tag{3.121} \]

and that withdrawals only happen on times \( \tau_j, \tau_{j+1}, ..., \tau_{K-1} \):

- Under static withdrawals

\[ V(W, A, \tau_{k+1}) = V(\max(W - \gamma_k^s, 0), A - \gamma_k^s, \tau_k) + f(\gamma_k^s), k = j, ..., K - 1, \tag{3.122} \]

- Under dynamic withdrawals

\[ V(W, A, \tau_{k+1}) = \sup_{\gamma_k^d \in [0, A]} [V(\max(W - \gamma_k^d, 0), A - \gamma_k^d, \tau_k) + f(\gamma_k^d)], k = j, ..., K - 1. \tag{3.123} \]

Let us denote the value of the variable annuity at inception, for an individual who dies within the time interval \([\tau_j, \tau_{j+1})\) as \( V_j(W, A, \tau = T) \). Then the value of the variable annuity, accounting for mortality is given by

\[ V_{\text{mort}}(W, A, \tau = T) = \sum_{j=0}^{T-1} (S(0, T - \tau_j) - S(0, T - \tau_{j+1}))V_j(W, A, \tau = T) \]

\[ + S(0, T)V_{\text{no-mort}}(W, A, \tau = T), \tag{3.124} \]

53
that is, after transforming back from $\tau$ to $t$, the value of the contract is

$$V_{\text{mort}}(W, A, t = 0) = \sum_{j=0}^{T-1} (S(0, t_{j+1}) - S(0, t_j))V_j(W, A, t = 0)$$

$$+ S(0, T)V_{\text{no-mort}}(W, A, t = 0), \quad (3.125)$$

where $S(0, t)$ is the survival probability that a person survives for the next $t$ years; $V_{\text{mort}}$ and $V_{\text{no-mort}}$ are the value of the variable annuity including and excluding mortality, respectively. The no-arbitrage fee $\alpha_g$ is then iteratively determined so that the contract value $V$ at inception ($t = 0$), is equal to the initial premium $W(0)$ paid by the investor (Milevsky and Salisbury, 2006, Dai et al., 2008, Chen et al., 2008). Note that we price $V$ under real-world probability measure, so $S(t, T)$ also needs to be described under the real world measure. However, in Section 3.1, there was no premium for mortality risk, so $S(t, T)$ is the same under both real-world measure and risk-neutral measure.

### 3.5 Hedging of GMBs

Often, it is inadequate to just have the correct pricing mechanism. Insurers prefer to reduce the risks in their products sold so that the required capital reserves can be lowered. This means that the insurer needs a way to deal with the stochastic mortality, interest rate and equity risk, embedded in the sales of GMBs. By hedging the risks, the insurer is able to control his risk profile and manipulate it to the level he desires. Unfortunately, since GMBs are long-term contracts, dynamic hedging strategies which are often proposed in the literature are not feasible due to the large transaction costs they would incur. Instead, we choose to demonstrate the static hedging strategy, that is, when the insurer hedges his position only at inception. A semi-static strategy can also be executed, i.e. the insurer rebalances his position at specific points in time.

#### 3.5.1 Evaluating the Greeks

In this section, we outline the Greeks (risk sensitivities) of GMBs in VAs. We choose to demonstrate this with the GMMB as it is one of the main building blocks for the other GMBs. Equation (3.88) shows that the GMIB can be represented as a sum of multiple GMMBs with varying maturities. We also do not investigate hedging for GMDBs as the value of GMDBs as a pre-retirement product is significantly lower than the living benefits. This is because the probability that a 50-year old dies before the retirement age is much lower than the survival probability, as evident from Figure 3.2.

We extend the usage of the FST algorithm in Jackson et al. (2008) to incorporate computation
of Greeks as suggested by Surkov and Davison (2010). Although Surkov and Davison (2010) initially describe the extension for jump models, we extend their methodology to the RSLN model. The Greek-FST method can be implemented at low extra computation cost, especially after the pricing of options (Surkov and Davison, 2010).

From Equation (3.86), we represent the GMMB as the product between a mortality component, \( \text{Mort}(T) \), and a financial component, \( \text{Fin}(T) \). Thus, when differentiating \( V_M(t, T, G) \), we can separate the set of Greeks by, differentiating with respect to mortality parameters, \( \theta_{\text{Mort}} \), and differentiating with respect to financial parameters \( \theta_{\text{Fin}} \). Thus, we obtain

\[
\frac{\partial V_M(t, T, G)}{\partial \theta_{\text{Mort}}} = \text{Mort}(t, T) \frac{\partial \text{Fin}(t, T, G)}{\partial \theta_{\text{Fin}}}
\]

(3.126)

\[
\frac{\partial V_M(t, T, G)}{\partial \theta_{\text{Fin}}} = \text{Mort}(t, T) \frac{\partial \text{Fin}(t, T, G)}{\partial \theta_{\text{Fin}}}
\]

(3.127)

In pricing, we split the GMMB into three distinct values, the European call, the Zero-Coupon Bond and the Mortality factor. In the following, we evaluate the Greeks in a similar fashion, that is, we separate the set of Greeks into financial Greeks and mortality Greeks.

1. Evaluation of the Financial Greeks

We begin with computing the Greeks with respect to the European call. First, using Equations (3.49) and (3.76), we can write

\[
\frac{\partial v}{\partial y}(y, t) = \mathcal{G}^{-1} \left[ iu \cdot e^{\Psi(u)(T-t)} \cdot \mathcal{G}[v(y, T)](u)1 \right]
\]

(3.129)

and

\[
\frac{\partial^2 v}{\partial y^2}(y, t) = \mathcal{G}^{-1} \left[ -u^2 \cdot e^{\Psi(u)(T-t)} \cdot \mathcal{G}[v(y, T)](u)1 \right].
\]

(3.130)
Thus, the delta of a European call, \( \frac{\partial C}{\partial F} \), is obtained via the chain rule

\[
\frac{\partial}{\partial F} = \frac{\partial}{\partial y} \frac{1}{F}.
\]

(3.131)

\[
\frac{\partial v}{\partial F}(y,t) = G^{-1} \left[ iu \cdot e^{\Psi(u)(T-t)} \cdot G[y(T)](u) \right] \cdot F.
\]

(3.132)

\[
\frac{\partial C}{\partial F}(F,t,T) = \pi_t \frac{G^{-1} \left[ iu \cdot e^{\Psi(u)(T-t)} \cdot G[\max(F(t)e^{Y(T)-Y(t)} - G, 0)](u) \right]}{F(T)}.
\]

(3.133)

Similarly, the gamma of a European call, \( \frac{\partial^2 C}{\partial F^2} \), is also obtained by the chain rule

\[
\frac{\partial^2}{\partial F^2} = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial F} \frac{1}{F} \right) = -\frac{1}{F^2} \frac{\partial}{\partial y} + \frac{1}{F^2} \frac{\partial^2}{\partial y^2}.
\]

(3.134)

\[
\frac{\partial^2 v}{\partial F^2}(y,t) = G^{-1} \left[ -(iu + u^2) \cdot e^{\Psi(u)(T-t)} \cdot G[y(T)](u) \right] \cdot F^2.
\]

(3.135)

\[
\frac{\partial^2 C}{\partial F^2}(F,t,T) = \pi_t \frac{G^{-1} \left[ -(iu + u^2) \cdot e^{\Psi(u)(T-t)} \cdot G[\max(F(t)e^{Y(T)-Y(t)} - G, 0)](u) \right]}{F^2(T)}.
\]

(3.136)

Note that the payoff function for a European call, \( \max(F(t)e^y - G, 0) \), does not depend on the interest rate or volatility parameters. Now, for any given parameter \( m \), where the option payoff does not depend on \( m \), the associated Greek requires just an extra multiplicative factor, \( M \), where

\[
M := \frac{\partial}{\partial m} \Psi(u)(T-t),
\]

(3.137)

in front of the Fourier transform of the option values. We can write

\[
\frac{\partial v}{\partial m}(y,t) = G^{-1} \left[ \frac{\partial}{\partial m} \left( e^{\Psi(u)(T-t)} \right) \cdot G[y(T)](u) \right]
\]

\[
= G^{-1} \left[ \frac{\partial}{\partial m} \Psi(u)(T-t) \cdot e^{\Psi(u)(T-t)} \cdot G[y(T)](u) \right]
\]

\[
= G^{-1} \left[ M \cdot e^{\Psi(u)(T-t)} \cdot G[y(T)](u) \right].
\]

(3.138)

\[
\frac{\partial C}{\partial m}(F,t,T) = \pi_t G^{-1} \left[ M \cdot e^{\Psi(u)(T-t)} \cdot G[\max(F(t)e^{Y(T)-Y(t)} - G, 0)](u) \right].
\]

(3.139)
A list of the multiplicative factors for the parameters which characterise each regime \((\kappa, \sigma, r)\) are provided in Table 3.3. Note that from Equation (3.42), the dynamics of the fund \(F\) are independent of \(\kappa\) under the risk-neutral measure and thus, the risk-neutral valuation of European calls written on \(F\) also do not depend on \(\kappa\). Hence, the derivatives of \(C(F, t, T)\) with respect to all \(\kappa\) parameters are 0, and are omitted from Table 3.3.

Table 3.3: Multiplicative Factors for \(r_1, r_2, \sigma_1, \sigma_2\)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility (\sigma_1)</td>
<td>((-iu + u^2)\sigma_1(T - t))</td>
</tr>
<tr>
<td>Volatility (\sigma_2)</td>
<td>(0)</td>
</tr>
<tr>
<td>Force of Interest (r_1)</td>
<td>((iu - 1)(T - t))</td>
</tr>
<tr>
<td>Force of Interest (r_2)</td>
<td>((iu - 1)(T - t))</td>
</tr>
</tbody>
</table>

This completes the derivations of the Greeks for the European call. Next we have to compute the financial Greeks for the zero-coupon bond. Since the bond value is only dependent on the risk-free force of interest in each of the regimes, there are only Greeks with respect to \(r_1\) and \(r_2\). After differentiating Equation (3.97) with respect to \(r_1\) and \(r_2\), we obtain

\[
\frac{\partial B}{\partial r_1}(t, T) = -e^{-r_2(T-t)}\pi'_t(T-t)D e^{(T-t)(A-(r_1-r_2)D)} 1, \tag{3.140}
\]

\[
\frac{\partial B}{\partial r_2}(t, T) = e^{-r_2(T-t)}\pi'_t(T-t)D e^{(T-t)(A-(r_1-r_2)D)} 1 - (T-t)e^{-r_2(T-t)}\pi'_t e^{(T-t)(A-(r_1-r_2)D)} 1. \tag{3.141}
\]

Hence, the financial Greeks for GMMBs can be easily evaluated at time \(t\) as \(Fin(t, T, G)\) comprises of only two financial components, the European call and the zero-coupon bond. A list of the Greeks is given below:

\[
\frac{\partial V_M}{\partial F}(t, T, G) = S(t, T)\frac{\partial C}{\partial F}(F, t, T)\tag{3.142}
\]

\[
\frac{\partial^2 V_M}{\partial F^2}(t, T, G) = S(t, T)\frac{\partial^2 C}{\partial F^2}(F, t, T)\tag{3.143}
\]

\[
\frac{\partial V_M}{\partial r_1}(t, T, G) = S(t, T)\left(\frac{\partial C}{\partial r_1}(F, t, T) + G\frac{\partial B}{\partial r_1}(t, T)\right)\tag{3.144}
\]

\[
\frac{\partial V_M}{\partial r_2}(t, T, G) = S(t, T)\left(\frac{\partial C}{\partial r_2}(F, t, T) + G\frac{\partial B}{\partial r_2}(t, T)\right)\tag{3.145}
\]

\[
\frac{\partial V_M}{\partial \sigma_1}(t, T, G) = S(t, T)\frac{\partial C}{\partial \sigma_1}(F, t, T)\tag{3.146}
\]
\[
\frac{\partial V_M}{\partial \sigma_2}(t, T, G) = S(t, T) \frac{\partial C}{\partial \sigma_2}(F, t, T).
\] (3.147)

2. Evaluation of the Mortality Greeks

Here we compute the Greeks with respect to mortality parameters. Taking the first and the second derivatives of Equation (3.5) with respect to \(\zeta_1\) and \(\zeta_2\), we obtain

\[
\frac{\partial S}{\partial \zeta_1}(t, T) = -C_1(t, T)S(t, T),
\] (3.148)
\[
\frac{\partial S}{\partial \zeta_2}(t, T) = -C_2(t, T)S(t, T),
\] (3.149)
\[
\frac{\partial^2 S}{\partial \zeta_1^2}(t, T) = C_1^2(t, T)S(t, T),
\] (3.150)
\[
\frac{\partial^2 S}{\partial \zeta_2^2}(t, T) = C_2^2(t, T)S(t, T).
\] (3.151)

Thus, using the results from Equations (3.148)-(3.151), the mortality Greeks for the GMMB are as follows:

\[
\frac{\partial V_M}{\partial \zeta_1}(t, T, G) = -C_1(t, T)S(t, T)(GB(t, T) + C(F, t, T))
\]
\[
= -C_1(t, T)V_M(t, T, G),
\] (3.152)
\[
\frac{\partial V_M}{\partial \zeta_2}(t, T, G) = -C_2(t, T)V_M(t, T, G),
\] (3.153)
\[
\frac{\partial^2 V_M}{\partial \zeta_1^2}(t, T, G) = C_1^2(t, T)V_M(t, T, G),
\] (3.154)
\[
\frac{\partial^2 V_M}{\partial \zeta_2^2}(t, T, G) = C_2^2(t, T)V_M(t, T, G).
\] (3.155)

This completes the computation of mortality and financial Greeks of the GMMB.

3.5.2 Static Hedging strategy

After finding the Greeks for the GMMB, we are able to outline a static hedging strategy and analyse its effectiveness. Suppose we wish to hedge a GMMB which matures at time \(T\), i.e. \(V_M(0, T, G)\). For the static hedge, we construct a hedging portfolio at time 0 that is held until maturity \(T\) without any rebalancing. Note that the methodology presented here is general enough to be applicable to other types of GMBs. Consider a hedging portfolio made up of various other GMMBs with differing maturities, \(t_j\), and different units, \(n_j\) assigned to each
GMMB. At time, \( t \), its value can be written as

\[
\Pi(t) = V_M(t, T, G) + \sum_{j=1}^{N} n_j V_M(t, t_j, G).
\]  

(3.156)

We require the portfolio to be immunised against changes in the financial and mortality parameters at time 0. Thus the system of equations we need to solve is given by

\begin{align*}
0 &= \Pi(0) = V_M(0, T, G) + \sum_{j=1}^{N} n_j V_M(0, t_j, G) \quad (3.157) \\
0 &= \frac{\partial \Pi}{\partial \zeta_1} = \frac{\partial V_M(0, T, G)}{\partial \zeta_1} + \sum_{j=1}^{N} n_j \frac{\partial V_M(0, t_j, G)}{\partial \zeta_1} \quad (3.158) \\
0 &= \frac{\partial \Pi}{\partial \zeta_2} = \frac{\partial V_M(0, T, G)}{\partial \zeta_2} + \sum_{j=1}^{N} n_j \frac{\partial V_M(0, t_j, G)}{\partial \zeta_2} \quad (3.159) \\
0 &= \frac{\partial^2 \Pi}{\partial \zeta_1^2} = \frac{\partial^2 V_M(0, T, G)}{\partial \zeta_1^2} + \sum_{j=1}^{N} n_j \frac{\partial^2 V_M(0, t_j, G)}{\partial \zeta_1^2} \quad (3.160) \\
0 &= \frac{\partial^2 \Pi}{\partial \zeta_2^2} = \frac{\partial^2 V_M(0, T, G)}{\partial \zeta_2^2} + \sum_{j=1}^{N} n_j \frac{\partial^2 V_M(0, t_j, G)}{\partial \zeta_2^2} \quad (3.161) \\
0 &= \frac{\partial \Pi}{\partial F} = \frac{\partial V_M(0, T, G)}{\partial F} + \sum_{j=1}^{N} n_j \frac{\partial V_M(0, t_j, G)}{\partial F} \quad (3.162) \\
0 &= \frac{\partial^2 \Pi}{\partial F^2} = \frac{\partial^2 V_M(0, T, G)}{\partial F^2} + \sum_{j=1}^{N} n_j \frac{\partial^2 V_M(0, t_j, G)}{\partial F^2} \quad (3.163) \\
0 &= \frac{\partial \Pi}{\partial r_1} = \frac{\partial V_M(0, T, G)}{\partial r_1} + \sum_{j=1}^{N} n_j \frac{\partial V_M(0, t_j, G)}{\partial r_1} \quad (3.164) \\
0 &= \frac{\partial \Pi}{\partial r_2} = \frac{\partial V_M(0, T, G)}{\partial r_2} + \sum_{j=1}^{N} n_j \frac{\partial V_M(0, t_j, G)}{\partial r_2} \quad (3.165) \\
0 &= \frac{\partial \Pi}{\partial \sigma_1} = \frac{\partial V_M(0, T, G)}{\partial \sigma_1} + \sum_{j=1}^{N} n_j \frac{\partial V_M(0, t_j, G)}{\partial \sigma_1} \quad (3.166) \\
0 &= \frac{\partial \Pi}{\partial \sigma_2} = \frac{\partial V_M(0, T, G)}{\partial \sigma_2} + \sum_{j=1}^{N} n_j \frac{\partial V_M(0, t_j, G)}{\partial \sigma_2}. \quad (3.167)
\end{align*}

We require \( N = 11 \), otherwise the system of equations is either overdetermined or underdetermined. Using matrix algebra, one can obtain the values of \( n_j \), for \( j = 1, 2, \ldots, 11 \). It is also advised that \( t_j \) are chosen to be a mix of both, longer and shorter maturities than \( T \), so that the duration and the convexity of the portfolio is close to that of the GMMB being hedged.
There is a concern on whether the insurer is allowed to short GMMBs in his hedging portfolio, i.e. whether one can allow \( n_j < 0 \). Thus, we outline an alternative hedging portfolio made up of simpler assets, similar to the one discussed in Luciano et al. (2011). We construct our hedging portfolio by purchasing \( N_{\text{Mort}} \) differing mortality derivatives, \( N_{\text{ZCB}} \) zero-coupon bonds and \( N_{\text{Call}} \) call options with various times to maturity \( t_j \).

\[
\tilde{\Pi}(t) = V_M(t, T, G) + \sum_{j=1}^{N_{\text{Mort}}} n_j \tilde{B}(t, t_j) + \sum_{j=N_{\text{Mort}}+1}^{N_{\text{Mort}}+N_{\text{Call}}} n_j C(F, t, t_j) + \sum_{j=N_{\text{Mort}}+N_{\text{Call}}+1}^{N} B(t, t_j),
\]

(3.168)

where \( \tilde{B}(t, T) \) is a q-Forward (the mortality derivative of choice) which is valued through

\[
\tilde{B}(t, T) = S(t, T)B(t, T).
\]

(3.169)

Since there are four mortality hedges, we require at least four different mortality-linked derivatives in our hedging portfolio. Similarly, we also require at least four European calls with different maturities in order to be able to solve for Delta, Gamma and Vegas being 0. Thus we choose four q-Forwards, three zero-coupon bonds and four European call to form our static hedging portfolio and obtain

\[
0 = \tilde{\Pi}(0) = V_M(0, T, G) + \sum_{j=1}^{4} n_j \tilde{B}(0, t_j) + \sum_{j=5}^{8} n_j C(F, 0, t_j) + \sum_{j=9}^{11} n_j B(0, t_j)
\]

(3.170)

\[
0 = \frac{\partial \tilde{\Pi}}{\partial \zeta_1} = \frac{\partial V_M(0, T, G)}{\partial \zeta_1} + \sum_{j=1}^{4} n_j \frac{\partial \tilde{B}(0, t_j)}{\partial \zeta_1}
\]

(3.171)

\[
0 = \frac{\partial \tilde{\Pi}}{\partial \zeta_2} = \frac{\partial V_M(0, T, G)}{\partial \zeta_2} + \sum_{j=1}^{4} n_j \frac{\partial \tilde{B}(0, t_j)}{\partial \zeta_2}
\]

(3.172)

\[
0 = \frac{\partial^2 \tilde{\Pi}}{\partial \zeta_1^2} = \frac{\partial^2 V_M(0, T, G)}{\partial \zeta_1^2} + \sum_{j=1}^{4} n_j \frac{\partial^2 B(0, t_j)}{\partial \zeta_1^2}
\]

(3.173)

\[
0 = \frac{\partial^2 \tilde{\Pi}}{\partial \zeta_2^2} = \frac{\partial^2 V_M(0, T, G)}{\partial \zeta_2^2} + \sum_{j=1}^{4} n_j \frac{\partial^2 B(0, t_j)}{\partial \zeta_2^2}
\]

(3.174)

\[
0 = \frac{\partial \tilde{\Pi}}{\partial F} = \frac{\partial V_M(0, T, G)}{\partial F} + \sum_{j=5}^{8} n_j \frac{\partial C(F, 0, t_j)}{\partial F}
\]

(3.175)

\[
0 = \frac{\partial^2 \tilde{\Pi}}{\partial F^2} = \frac{\partial^2 V_M(0, T, G)}{\partial F^2} + \sum_{j=5}^{8} n_j \frac{\partial^2 C(F, 0, t_j)}{\partial F^2}
\]

(3.176)
Similar to the hedging portfolio made up of GMMBs, we utilise matrix algebra to solve for \( n_j \), the number of units needed for each asset in the hedging portfolio.

### 3.5.3 Semi-static Hedging

For the semi-static hedge, we rebalance our portfolio at specific time points \( t_i \) where \( 0 = t_1 < t_2 < \ldots < t_i < \ldots < T \). The procedure can be outlined as follows:

1. Construct the static hedging portfolio \( \Pi(0) \) at time 0 as specified in Section 3.5.2.
2. Evaluate \( \Pi(t_2) \) at the first rebalancing time \( t_2 \).
3. Construct a new static-hedging portfolio to hedge a guarantee \( V_M(0, T-t_2, G) \), which has incorporated all the new information from time \( t_1 \) to \( t_2 \).
4. Change the number of units in the hedging instruments if the change in \( n_j \) is greater than \( \alpha_{tol} \) where \( \alpha_{tol} \) is some tolerance level. This stops any unnecessary transaction costs.
5. Repeat steps 2-4 for all time points \( t_i \) up until \( T \).
Chapter 4

Numerical Results

In this chapter we provide some pricing and hedging results for the various GMBs. We first start with the pricing results for path-independent GMBs in Section 4.1, then hedging results in Section 4.3 and finally provide GMWB results in Section 4.2. For the whole of this chapter, we use the parameters in Tables 4.1 and 4.2 to describe our financial and mortality model, unless otherwise specified.

Table 4.1: Parameters used for the RSLN model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Fund Value $F(0)$</td>
<td>100</td>
</tr>
<tr>
<td>Initial Probability of being in Regime 1 $p$</td>
<td>0.9999</td>
</tr>
</tbody>
</table>

**Regime 1**

- Drift $\kappa_1$ : 0.10
- Volatility $\sigma_1$ : 0.15
- Force of Interest $r_1$ : 0.05
- Transition Rate $a_{12}$ : 0.40

**Regime 2**

- Drift $\kappa_2$ : 0.04
- Volatility $\sigma_2$ : 0.40
- Force of Interest $r_2$ : 0.02
- Transition Rate $a_{21}$ : 0.30

Table 4.2: Parameters used for the 2-factor independent ATSM

<table>
<thead>
<tr>
<th>$\delta_1$</th>
<th>$\delta_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.1004$</td>
<td>$-0.1347$</td>
<td>$1.4285 \times 10^{-4}$</td>
<td>$4.9659 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
The mortality parameters in Table 4.2 are estimated through the 2-factor model fitted to Australian 50 year old male mortality data provided in the Human Mortality Database. We have chosen these particular financial parameters to demonstrate two distinctly different regimes, one with high volatility and low returns, and the other with low volatility and high returns. The two regimes represent the “bear” and “bull” market, respectively. Furthermore, since the Markov chain $X$ which modulates the regime switches is observable, then we are almost certain of which regime we are currently residing in, i.e. $p$, the initial probability needs to be very close to either 0 or 1.

4.1 Price sensitivity analysis for GMBs

In this section we investigate the price sensitivity of GMMBs, GMIBs and GMDBs to the model parameters. For illustrative purposes, we assume that the nominal values for the guarantees in GMBs is $100, i.e. G = 100. However, this means that the GMIB provides a guaranteed benefit of $T \times G$ over the span of its contract. We scale the value of GMIBs, $V_I(t, T, G)$ by $\frac{1}{T}$ so that the total guaranteed benefit is the same as in the case of the GMMB and GMDB, and that comparisons can be made between the GMBs.

4.1.1 The effect of the guarantee level on GMMB valuation

First, we investigate the sensitivity of the price of GMMBs to various different guarantee levels $G$. We expect the price of the GMMB to increase as $G$ increases as the policyholder is guaranteed more. Figure 4.1 plots the change in prices as $G$ varies, for GMMBs with different contract lengths. The results confirm that prices increase with the guarantee levels, as well depicting an increasing convexity in the price function as the length of the contract shortens. The convexity in prices is not surprising as from Equation (3.86), the GMMB consists of a European call, and similar price convexity is seen when plotting European calls against various strike levels. Furthermore, we can see that for contracts lasting 10 years or less, a guarantee level of 50, which is half the original starting fund level, has almost no value, since the GMMB’s value is equal to the starting fund (recalling that in the parameters for our financial model, $F(0) = 100$).

4.1.2 The effect of mortality on GMMB valuation

In Figure 4.2, we plot the value of the GMMB priced with and without the mortality factor. It is straightforward to see that the mortality experience provides a significant change in the fair value of the GMMB. The difference between the two lines can be interpreted as a discounting mortality factor. From the dashed line, we can observe that the price of the GMMB decreases as the time to maturity increases beyond 8 years. The price increase reflects the general increase
in value of guaranteed benefits further along in the future which compensate for the increasing uncertainties; the price decrease is caused by the increasing mortality which discounts the benefit since GMMBs only provide benefits if the policyholder is alive. This is in contrast to the solid line, where the mortality factor is not taken into consideration. The kink seen at the end of the solid line is a product of the large difference in volatilities and interest rates between the regimes for the financial model. We choose to keep the large difference in parameters between the two regimes to indicate the contrast between the two and later test the price sensitivity to regime parameters in Section 4.1.4.

4.1.3 The effect of maturity dates on GMB valuation

Figure 4.3 plots the prices of GMMBs, GMIBs and GMDBs across increasing maturities. As noted in Figure 4.2, the value of GMMBs starts to decrease after 8 years, and continues to decrease at an increasing rate, due to the mortality discounting factor becoming more significant. In contrast, the GMDB starts off at a very low value (due to the unlikelihood of death in the early ages) and grows quickly with increasing maturity. The rapid increase in the GMDB value accounts for both the increase in death probability as well as the increasing time to maturity in the financial option. Since, we have scaled the price of the GMIB down by the number of payments, the price of the GMIB is equal to the average of the all the GMMBs with shorter maturities. Thus, the GMIB displays similar characteristics to the GMMB and the decrease in value as $T$ increases is also seen in GMIBs, albeit with some lag.
The results also reinforce the claims made in Section 3.5.1, that the GMDB is much cheaper than the GMMB and GMIB for contracts which mature in 15 years, i.e. the retirement age 65. However, as we can see in Figure 4.3, GMDBs do have significant value when issued for longer periods, e.g. for life such as in GLWBs.

4.1.4 Price sensitivity to regime parameters

Lastly, we plot the price sensitivity of GMMBs to the parameters set in the RSLN model since the financial model is not calibrated to any empirical data. The risk-neutral valuation of GMMBs only depends on the risk-free rates, volatilities, transition rates and initial probability. For our sensitivity analysis, we alter the risk-free force of interest, and the volatility of Regime 2, the “bear” regime. Figure 4.4 provides a surface plot of the prices of GMMBs with $T = 15$. It is straightforward to see that the price increases with increasing volatility and decreasing interest rates. This can be intuitively reasoned as the value of a minimum investment return becomes more valuable if the future carries more uncertainties. Furthermore, increasing interest rates discount the guaranteed benefit more heavily, which lowers its present day value. A percentage change in the force of interest more significantly affects the price than a percentage change in volatility. We table the full results in Appendix D for a more complete look at the price sensitivities to regime parameters, across GMMBs, GMIBs and GMDBs as well as different maturity lengths. All the results show similar characteristics with the price increasing with
higher volatility, and decreasing with higher interest rates.

Figure 4.3: Price of GMMBs, GMIBs and GMDBs across various maturities

Figure 4.4: Price sensitivity of GMMBs to the parameters describing Regime 2
4.2 GMWB Pricing Results

In this section we outline some of the numerical pricing results for the GMWB using the FST algorithm. The prices (fee charged by the insurer) outlined in this section are in basis points (b.p.), where 1 basis point is equivalent to 0.01%. First, we compare our results to other results reported in the literature. Since the majority of the literature have priced GMWBs under the GBM, our regime-switching model can be reduced to the GBM model by allowing the parameters for the two regimes to be the same. Next, we provide the numerical results and sensitivity analysis of GMWB prices where the market is described by the two regimes, with parameters specified in Table 4.1 at the start of this chapter. We also require some other GMWB specific parameters which are documented in Table 4.4. Table 4.3 contains the parameters used for the GBM model in Section 4.2.1; the drift, volatility and force of interest under Regime 1 are the same as in Regime 2, which reduces the RSLN model to the GBM model.

<table>
<thead>
<tr>
<th>Table 4.3: Financial Parameters used for GMWBs under the GBM Model.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Fund Value</td>
</tr>
<tr>
<td>Initial Probability of being in Regime 1</td>
</tr>
<tr>
<td>Regime 1 Drift</td>
</tr>
<tr>
<td>Volatility</td>
</tr>
<tr>
<td>Force of Interest</td>
</tr>
<tr>
<td>Transition Rate</td>
</tr>
<tr>
<td>Regime 2 Drift</td>
</tr>
<tr>
<td>Volatility</td>
</tr>
<tr>
<td>Force of Interest</td>
</tr>
<tr>
<td>Transition Rate</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 4.4: GMWBs specific parameters.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
</tr>
<tr>
<td>Guarantee</td>
</tr>
<tr>
<td>Mutual fund fee</td>
</tr>
</tbody>
</table>

4.2.1 Comparison with results from the Existing Literature

We compare our results to those in Chen et al. (2008), Chen and Forsyth (2008) and Luo and Shevchenko (2014), who price the GMWB under the GBM using other algorithms such as finite difference schemes. In Tables 4.5 and 4.6, we present GMWB prices with 10-year maturities under the dynamic withdrawal case. The cost structure for overdrawing the guaranteed amount

67
is specified in Table 3.2. We find that our results are consistent with those found in Chen et al. (2008), where the only significant deviation is when α_m = 2%. Similarly, for Table 4.7, our pricing results for GMWBs are consistent with those generated by either Chen and Forsyth (2008) and Luo and Shevchenko (2014), which are tabled under CF and LS, respectively. Furthermore, we find that our results are also consistently slightly higher than those reported in existing literature, although this could be just due to rounding errors.

Table 4.5: Comparison of results: Varying sigma, α_m = 1%.

<table>
<thead>
<tr>
<th>σ</th>
<th>Chen et al. (2008) α_g</th>
<th>FST (Regime-Switch) α_g</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>117 b.p.</td>
<td>118 b.p.</td>
</tr>
<tr>
<td>0.2</td>
<td>214 b.p.</td>
<td>215 b.p.</td>
</tr>
<tr>
<td>0.25</td>
<td>326 b.p.</td>
<td>327 b.p.</td>
</tr>
<tr>
<td>0.3</td>
<td>440 b.p.</td>
<td>442 b.p.</td>
</tr>
<tr>
<td>0.35</td>
<td>552 b.p.</td>
<td>554 b.p.</td>
</tr>
</tbody>
</table>

Table 4.6: Comparison of results: Varying management fee, σ = 0.15.

<table>
<thead>
<tr>
<th>α_m</th>
<th>Chen et al. (2008) α_g</th>
<th>FST (Regime-Switch) α_g</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>88 b.p</td>
<td>88 b.p</td>
</tr>
<tr>
<td>0.5%</td>
<td>102 b.p</td>
<td>102 b.p</td>
</tr>
<tr>
<td>1%</td>
<td>117 b.p</td>
<td>118 b.p</td>
</tr>
<tr>
<td>1.5%</td>
<td>136 b.p</td>
<td>138 b.p</td>
</tr>
<tr>
<td>2%</td>
<td>157 b.p</td>
<td>161 b.p</td>
</tr>
</tbody>
</table>

Table 4.7: Comparison of results: Constant cost structure η = 10%.

<table>
<thead>
<tr>
<th>Volatility</th>
<th>CF</th>
<th>LS</th>
<th>FST</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>129.1 b.p</td>
<td>129.1 b.p</td>
<td>130.8 b.p</td>
</tr>
<tr>
<td>0.3</td>
<td>293.3 b.p</td>
<td>293.5 b.p</td>
<td>295.5 b.p</td>
</tr>
</tbody>
</table>

4.2.2 Pricing under a regime-switching model

Next, we report pricing results generated under the regime-switching model. Results for GMWBs under static withdrawal case are presented in Table 4.8 and the dynamic withdrawal case in Table 4.9. We also price the GMWB with and without mortality effects to observe the effect of incorporating mortality in GMWB prices. It is straightforward to see that as maturities of
GMWBs increase, there is more of a need to incorporate mortality into pricing. As for the 15-year maturity GMWBs sold to 50 year old male Australians as a pre-retirement product, there is almost no change in prices after incorporating mortality (3 basis points equates to 0.03%).

Table 4.8: Regime Switching fees across various $T$: Static withdrawals.

<table>
<thead>
<tr>
<th>$T$</th>
<th>Static without mortality</th>
<th>Static with mortality</th>
</tr>
</thead>
</table>

Table 4.9: Regime Switching fees across various $T$: Dynamic withdrawals.

<table>
<thead>
<tr>
<th>$T$</th>
<th>Dynamic without mortality</th>
<th>Dynamic with mortality</th>
</tr>
</thead>
</table>

Figure 4.5 further investigates the mortality effect on pricing. The fair fee charged for GMWBs under the static withdrawal cases is plotted, and we observe that after incorporating mortality, the fee increases across the board. However, as shown in Tables 4.8 and 4.9, the effect on prices is minimal for low values of $T$. Furthermore, the fair fee decreases to zero when $T$ increases since the guaranteed annual withdrawal $G = \frac{100}{T}$ decreases, which lowers the value of the GMWB rider. We observe that only when $T > 30$, that there is significant difference between pricing with and without mortality. However the time-frame which there is a difference between pricing with and without mortality is low; by $T = 35$, the fair fee charged is close to zero, i.e. the value of the GMWB rider is worthless. If $T > 35$, the fair fees become negative, which is be explained by low withdrawal rate $G = \frac{100}{35} = 2.85\%$. This is remarkably low considering that the risk-free force of interest is 5% and 2% in Regime 1 and Regime 2, respectively, thus causing the fair fee charged to be negative.

### 4.2.3 Price sensitivity to regime parameters

In Figure 4.6, we plot the change in fair fee charged for GMWB maturing in 15 years to changes in parameters in Regime 2. The surface has a similar shape to that reported in Figure 4.4: the price increases as volatility increases and decreases as the force of interest decreases. Additional results of GMWB price sensitivities to regime parameters are recorded in Appendix D.
Figure 4.5: Pricing GMWBs with and without mortality under static withdrawals.

Figure 4.6: Price sensitivity of GMWBs under no mortality and dynamic withdrawal case to regime parameters.
4.3 Hedging results

4.3.1 Greeks plots

After implementing the methodology described in Section 3.5.1, we are able to numerically compute and plot the Greeks for the GMMB as a function of underlying parameters. In Figure 4.7, (a) and (b) plot the delta and gamma of the GMMB. For a 15-year long GMMB, there is little change in the delta which is reflected in the linearity in prices that we observed in Figure 4.1. The delta and gamma of the GMMB with maturity of one year is more interesting. Here, the Greeks plots resemble those seen in a European call. As the stock price decreases to 0, the change in price also goes to 0, as the call option is almost certain to be exercised and the final payoff is the guarantee level. Similarly, if the stock increases to ∞, the change in price goes to 1, as the call option is almost certain to be not exercised and the final payoff is the stock price. Hence, a $1 change in stock price is almost equivalent to a $1 change in the GMMB price.

In Figure 4.7, (c) and (d) plot the percentage change in price due to a change in the volatilities in regimes 1 and 2. Similarly to vanilla European options, the value of the GMMB increases with rising volatility. Furthermore, we observe that the price of GMMBs with longer maturities rise at an increasing rate. In Figure 4.7, (e) and (f) plot the percentage change in price due to changes in the force of interest in regimes 1 and 2. In both plots, the values are negative; the risk-neutral price of the GMMB decreases with higher risk-free rates due to higher discounting. It is also observed that in all four plots, Figure 4.7(c)-(f), GMMBs with longer maturities are more sensitive to changes in interest rate and market volatility.

Figure 4.8 shows the GMMB price sensitivity with respect to the mortality factors ζ_1 and ζ_2 across time. Since ζ_1 and ζ_2 are unobserved, there isn’t any meaningful interpretation from these graphs. However, the values in (a) are consistently lower than those in (b), which signifies that ζ_2 has more importance than ζ_1 when determining the mortality rate μ(t). This is reinforced in (c) and (d) which show the second derivative with respect to these mortality latent variables.

4.3.2 Static Hedging Portfolio

In this section we assume that the insurer wishes to hedge a GMMB he has sold, which matures in 15 years. We solve for the static hedging portfolio with Equations (3.156) and (3.168), and report the weights in Table 4.10. Portfolio 1 is made up of various GMMBs with different maturities as solved from Equation (3.156). However, if a constraint that GMMBs cannot be shorted is applied, we can create an alternate portfolio, Portfolio 2, which is made up of various simpler derivatives - European calls, Zero-Coupon Bonds (ZCB) and q-Forwards. These results are then used to create the static hedging portfolio at $t = 0$. We test the hedge effectiveness by
Figure 4.7: Financial Greeks for GMMB $t = 0$, $G = 100$

comparing the loss distributions and various statistics throughout time.
4.3.3 Profit and loss distribution

After running 500 simulations, we report the static hedging performance in Table 4.11. It is surprising to see the Portfolio 2, which is made up of a collection of derivatives, to have a larger standard deviation than the unhedged portfolio. However Portfolio 1 does provide a reduction in standard deviation compared to the unhedged portfolio. This reduction is not very large which leads us to conclude that the static hedging strategy can be improved through additional rebalancing, i.e. semi-static hedging strategies. Both portfolios demonstrate lower 90% quantile of losses, indicating that the static hedge reduces tail risks and provides some benefit. This motivates us to investigate the performance of the static hedge throughout time, and in particular, recognise the scenarios when the static hedge becomes ineffective.

Table 4.12 provides the mean and standard deviation of the three portfolios throughout time, at least, until the GMMB sold is expires. We assume that the portfolios are unwound upon the
Table 4.10: Assets used in the Static Hedging Portfolios for GMMB

<table>
<thead>
<tr>
<th>Time ($t_j$)</th>
<th>Portfolio 1</th>
<th>Portfolio 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Type</td>
<td>Price ($)</td>
</tr>
<tr>
<td>9</td>
<td>GMMB</td>
<td>114.5850</td>
</tr>
<tr>
<td>10</td>
<td>GMMB</td>
<td>114.4044</td>
</tr>
<tr>
<td>11</td>
<td>GMMB</td>
<td>114.0672</td>
</tr>
<tr>
<td>12</td>
<td>GMMB</td>
<td>113.5903</td>
</tr>
<tr>
<td>13</td>
<td>GMMB</td>
<td>112.9863</td>
</tr>
<tr>
<td>14</td>
<td>GMMB</td>
<td>112.2653</td>
</tr>
<tr>
<td>15</td>
<td>GMMB</td>
<td>111.4354</td>
</tr>
<tr>
<td>16</td>
<td>GMMB</td>
<td>110.5039</td>
</tr>
<tr>
<td>17</td>
<td>GMMB</td>
<td>109.4773</td>
</tr>
<tr>
<td>18</td>
<td>GMMB</td>
<td>108.3621</td>
</tr>
<tr>
<td>19</td>
<td>GMMB</td>
<td>107.1644</td>
</tr>
<tr>
<td>20</td>
<td>GMMB</td>
<td>105.8901</td>
</tr>
</tbody>
</table>

1The standard deviations are presented in Figure 4.9. Static hedging portfolio performs very well up until when some of the assets in the portfolios expire, which are then reinvested into risk-free assets. The jump in standard deviation from year 10 to 11 corresponds to the point in time when a large number of European calls expire. Similarly, in Portfolio 1, there is a large number of 12, 13, 14 year GMMBs, which when expire, cause larger increases in standard deviation. Thus, it would be interesting to investigate how the standard deviation of the portfolio evolves over time, if every asset class in the portfolio was restricted in number of units. We leave this analysis for future research.

It is also clear that Portfolio 1 performs significantly better than Portfolio 2. Portfolio 1 also has lower standard deviation than the unhedged portfolio, showing that static hedging is still beneficial to the insurer. From the results, it seems that the static hedge should be rebalanced after 10 years. Figure 4.10 display the histograms of losses after 15 years for the three portfolios. From the figures it is evident that for an unhedged portfolio, the insurer faces unlimited liability and limited profit whereas, after hedging, the distribution of losses becomes more like the normal distribution.

Table 4.11: Static Hedging Results at maturity

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Mean</th>
<th>Std Dev</th>
<th>90% Quantile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unhedged</td>
<td>97.5658</td>
<td>402.5393</td>
<td>407.9298</td>
</tr>
<tr>
<td>Hedge Portfolio 1</td>
<td>2.5680</td>
<td>301.4457</td>
<td>223.2447</td>
</tr>
<tr>
<td>Hedge Portfolio 2</td>
<td>66.3031</td>
<td>575.9141</td>
<td>382.2526</td>
</tr>
</tbody>
</table>

1We make the assumption that GMMBs can be shorted again in Portfolio 1 to unwind GMMBs whose maturities are longer than 15 years.
Table 4.12: Static Hedging Results over time

<table>
<thead>
<tr>
<th>Time</th>
<th>Portfolio 1</th>
<th>Portfolio 2</th>
<th>Unhedged</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std Dev</td>
<td>Mean</td>
</tr>
<tr>
<td>1</td>
<td>-8.1858×10^-6</td>
<td>5.0151×10^-6</td>
<td>0.0536</td>
</tr>
<tr>
<td>2</td>
<td>8.4889×10^-6</td>
<td>5.1694×10^-6</td>
<td>0.0598</td>
</tr>
<tr>
<td>3</td>
<td>-2.7127×10^-6</td>
<td>3.3304×10^-6</td>
<td>0.0618</td>
</tr>
<tr>
<td>4</td>
<td>1.6217×10^-6</td>
<td>3.8073×10^-6</td>
<td>0.0391</td>
</tr>
<tr>
<td>5</td>
<td>2.5152×10^-6</td>
<td>1.1018×10^-5</td>
<td>0.0669</td>
</tr>
<tr>
<td>6</td>
<td>3.3557×10^-7</td>
<td>3.6513×10^-5</td>
<td>0.0390</td>
</tr>
<tr>
<td>7</td>
<td>-8.2298×10^-6</td>
<td>1.5265×10^-4</td>
<td>-0.0697</td>
</tr>
<tr>
<td>8</td>
<td>-2.7026×10^-5</td>
<td>9.4881×10^-4</td>
<td>-0.2801</td>
</tr>
<tr>
<td>9</td>
<td>-3.2128×10^-4</td>
<td>0.0202</td>
<td>-0.4276</td>
</tr>
<tr>
<td>10</td>
<td>-0.1788</td>
<td>1.9637</td>
<td>-441.3486</td>
</tr>
<tr>
<td>11</td>
<td>0.3238</td>
<td>18.6402</td>
<td>76.3913</td>
</tr>
<tr>
<td>12</td>
<td>-1.8131</td>
<td>68.9695</td>
<td>77.1555</td>
</tr>
<tr>
<td>13</td>
<td>10.0063</td>
<td>165.4868</td>
<td>72.9699</td>
</tr>
<tr>
<td>14</td>
<td>-3.2639</td>
<td>274.2316</td>
<td>68.9279</td>
</tr>
<tr>
<td>15</td>
<td>2.5680</td>
<td>301.4457</td>
<td>66.3031</td>
</tr>
</tbody>
</table>

Figure 4.9: Standard deviation over time for various hedging portfolios
Figure 4.10: Loss Histograms for Hedging Portfolios at $T = 15$
Chapter 5

Conclusion and Future Research

This thesis provides a general pricing framework for various types of GMBs under stochastic mortality, equity risk and interest rate risk. This is done by implementing two models in conjunction: a RSLN model is used to model the financial market and an affine term structure model is used to model the mortality dynamics. We provide price sensitivities of GMMBs, GMIBs, GMDBs and GMWBs sold to pre-retirement Australian males. We also demonstrate the construction of a static hedging portfolio for the sale of the GMMB, and show that a portfolio consisting solely of GMMBs with other maturities performs better than a hedge portfolio made up of various simpler assets, such as q-Forwards, zero-coupon bonds and European calls. To the best of our knowledge, this thesis also pioneers the usage of the FST algorithm in computing the Greeks of GMBs as well as the pricing of GMBs in a regime-switching environment.

5.1 Summary of results

This thesis provides pricing results for GMMBs, GMIBs, GMDBs and GMWBs as a pre-retirement product targeted towards 50-year-old Australian males.

- The fair value of a GMMB is increasing for increasing maturities for the first 8 years, before starting to decrease. The price increase reflects the general increase in value for guaranteed benefits further into the future, when there are more uncertainties; the price decrease is caused by the increasing mortality which discounts the value of the benefit since GMMBs can only be exercised if the policyholder is alive.

- Since the GMIB is the sum of multiple GMMBs with varying maturities, the increase then decrease in price as maturities increase is also observed for the GMIB.

- The value of a GMDB is negligible in the early years. However, its value quickly increases
as the length of the contract increases. Nevertheless, in the context of GMBs maturing at retirement age, i.e. 15-year long contracts, the value of the GMDB is close to zero, and is significantly smaller than its living benefit counterparts (GMMB and GMIB).

- Mortality has minimal effect on the price of 15-year long GMWBs regardless of the withdrawal type (dynamic or static). This is demonstrated through low mortality for GMWBs for short maturities and high guarantees, as the probability of death is very low. As maturities increase, mortality plays an increasing role on the prices; however, for GMWBs with longer maturities, the level of guarantee is lower, and thus the guaranteed withdrawals are not as valuable. Ultimately, the fair fee charged for GMWB approaches zero, regardless of whether mortality is included in the model. Under the static withdrawal case, only contracts lasting longer than 30 years are significantly affected by mortality. However, for contracts longer than 35 years, the value of the GMWB is worthless.

- There is a strong sensitivity of prices to regime-switching parameters, and a percentage increase in the force of interest increases the value by more than a decrease in percentage decrease in the volatility.

This thesis also provides static hedging analysis for GMMBs.

- A static hedging portfolio provides a reduction in tail probabilities, compared to an unhedged portfolio.

- However, a static hedging portfolio does not provide effective variance reduction compared to an unhedged portfolio. Thus, semi-static hedging strategies are recommended.

- The static hedging portfolio is effective up until two to three of the assets in the portfolio mature.

- A portfolio made up of various GMMBs performs better than a constructed from simpler assets, such as European calls, zero-coupon bonds and q-Forwards. This is consistent with the results from Coleman et al. (2006) and Coleman et al. (2007) who note that hedging with more complex derivatives is better than simpler ones (such as options is better than futures of the underlying fund).

5.2 Research contributions

This thesis contributes to the existing literature in several ways:

- This research provides a general pricing and hedging framework for various guarantees embedded in VAs, including GMMBs, GMIBs, GMDBs and GMWBs. This is numerically applied through a flexible and fast computational algorithm - the Fourier Space Time-stepping algorithm.
This research extends the pricing literature of GMBs under regime-switching models. This is one of the main differences between this research and past literature which consider pricing multiple GMBs (Bauer et al., 2008 and Bacinello et al. (2011)). The price sensitivities of GMBs to the regime-switching parameters are also evaluated.

This thesis also quantifies the effect of including mortality on pricing of GMBs. The stochastic mortality is incorporated into the pricing through an affine model described in Blackburn and Sherris (2013). It is shown that mortality is essential in pricing GMMBs, GMIBs and GMDBs. However, it is explicitly shown through numerical tests that there is little to no mortality effect on prices of GMWBs for short maturities. As for extremely long maturities, the GMWB rider is worthless.

It formally describes the application of the Greek FST methodology to compute the Greeks of GMBs under a regime-switching environment.

This research extends the literature on delta-gamma hedging mortality derivatives - GMMBs in particular. It provides extensive analysis on static hedging portfolios. A comparison of the hedging effectiveness is shown through the standard deviations throughout time as well as profit and loss distribution histograms.

5.3 Limitations and future areas of research

In this thesis we utilise several simplifying assumptions which may limit the applicability of the results produced. Possible future extensions and areas of research could address the following points:

- We assume there is no risk premium associated with a switch between the regimes. This stems from the usage of a single regime-switching model and can be relaxed via the use of a double regime-switching model such as the one outlined in Shen et al. (2014).

- We assume that under each regime, the financial market follows a GBM. A future area of research is to extend the financial model to have a more sophisticated model under each regime, e.g. a stochastic interest rate model instead of the constant force of interest assumed in the GBM. It may also be interesting to compare the pricing results to see if the added complexity provides any significant improvements.

- We assume that the Markov chain which controls the regime-switching process is observable. However, a hidden Markov chain is more realistic as it is unlikely that one knows exactly when a regime change has occurred. This will also require calibration to empirical data, which we have not included in this thesis.

- The guarantees are priced without any additional features such as roll-ups, deferrals or ratchets. This conflicts with what is currently being offered by insurers and bought by
policyholders. These various features can be factored in to provide a more wholesome and realistic pricing framework of GMBs.

- We assume that the policyholders do not surrender their policy by choice. As shown in Milevsky and Salisbury (2001), the policyholder has no incentive to continue holding a guarantee that is deep out-of-the-money. Thus, the policyholder is likely to surrender a guarantee that is very unlikely to be exercised, rather than continue to be charged a fee on investment returns. We do not account for this behaviour in our pricing framework.

- Similar to Blackburn and Sherris (2013) and Ziveyi et al. (2013), we also assume that there is no mortality risk premium in the mortality model.

- We do not value the GLWB guarantee in this research due to its non-fixed maturity date. However, GLWBs have high election rates (LIMRA (2013)) in practice and require attention in both pricing and hedging.

- Moreover, the mortality effect which was found to be ineffective in GMWBs is more likely to be a significant factor for GLWBs.

- It is likely that there is a more significant mortality effect on prices for GMWBs when issued to an older group, such as the 75+ year olds.

- We do not perform numerical investigation for the semi-static hedging portfolios. We also do not consider natural hedging strategies which can arise between GMIBs and GMDBs, similar to those seen in term assurance and annuities.

- We assume that q-Forwards and European calls of long maturities are readily tradeable without any additional cost. This however is not the case in reality as calls with longer maturities are rare. Currently there are no listed calls for the ASX index which last longer than one year in maturity. The longest calls for single stocks are up to three years, nowhere near to the 10+ year calls specified in the hedging portfolio.
Appendix A

Characteristic Function of the Discounted Log Return

In this section, we provide the proof of Equation (3.57), which is the discounted characteristic function of $Y$. Recall that $Y$ is the log return of the fund $F$ and from Equation (3.41):

$$dY(t) = \left( r(t) - \frac{1}{2} \sigma^2(t) \right) dt + \sigma(t) dZ(t). \quad (A.1)$$

First, let us denote the conditional and unconditional characteristic functions as $\phi_{Y(t)|F_X}$ and $\phi_{Y(t)}$, respectively:

$$\phi_{Y(t)|F_X}(0,t,u) := E^\theta[ e^{iuY(t)|F_X} ], \quad (A.2)$$
$$\phi_{Y(t)}(0,t,u) := E^\theta[ \phi_{Y(t)|F_X(t)}(0,t,u) ], \quad (A.3)$$

where

$$i = \sqrt{-1}. \quad (A.4)$$

We also require the discounted characteristic functions $\phi_{\tilde{Y}(t)|F_X(t)}$ and $\phi_{\tilde{Y}(t)}$, which is the goal of this proof:

$$\phi_{\tilde{Y}(t)|F_X(t)}(0,t,u) := e^{-\int_0^t r(s) ds} \phi_{Y(t)|F_X(t)}(0,t,u) \quad (A.5)$$
$$= e^{-R(t)} \phi_{Y(t)|F_X(t)}(0,t,u), \quad (A.6)$$
$$\phi_{\tilde{Y}(t)}(0,t,u) := E^\theta[ \phi_{\tilde{Y}(t)|F_X(t)}(0,t,u) ], \quad (A.7)$$
where

\[ R(t) := \int_0^t r(s)ds. \] (A.8)

Now we aim to find an expression for \( \tilde{\phi}_Y(t,0,t,u) \). Since \( X \) is Markovian, then the filtration \( \mathcal{F}^X(t) \) only requires \( X(t) \). Furthermore, Dufour and Elliott (1999) show the following representation of the chain \( X \) under the risk-neutral measure \( \mathbb{Q}^\theta \):

\[ X(t) = X(0) + \int_0^t AX(s)ds + M^\theta(t), \] (A.9)

where \( M^\theta(t) \) is a \((\mathcal{F}^X, \mathbb{Q}^\theta)\)-martingale. Recall that \( X(t) \) is a \( N \times 1 \) vector with the value 1 in the \( j \)-th element and zeroes elsewhere to signify regime \( j \). So we can rewrite equation (A.7) as

\[ \tilde{\phi}_Y(t,0,t,u) = E^{\theta}[\tilde{\phi}_Y(t)|\mathcal{F}^X(t)](0,t,u) = \langle E^{\theta}[X(t)\tilde{\phi}_Y(t)|\mathcal{F}^X(t)](0,t,u), 1 \rangle \] (A.10)

where

\[ h(t,u) := X(t)\tilde{\phi}_Y(t)|\mathcal{F}^X(t)(0,t,u). \] (A.12)

Let us consider \( e^{iuY} \). By Itô’s lemma, we obtain

\[ d\left(e^{iuY(t)}\right) = e^{iuY(t)}(iudY(t) - \frac{u^2}{2}(dY(t))^2) = e^{iuY(t)}\left(iu(r(t) - \frac{1}{2}\sigma^2(t))dt - \frac{u^2}{2}\sigma^2(t)dt + iu\sigma(t)dZ^\theta(t)\right). \] (A.13)

Thus, we can find \( d\tilde{\phi}_Y(t)|\mathcal{F}^X(t)(0,t,u) \) using equation (A.2) as follows:

\begin{align*}
\quad d\tilde{\phi}_Y(t)|\mathcal{F}^X(t)(0,t,u) &= dE^{\theta}[e^{iuY(t)}|\mathcal{F}^X(t)] \\
&= E^{\theta}[de^{iuY(t)}|\mathcal{F}^X(t)] \\
&= E^{\theta}[e^{iuY(t)}\left(iu(r(t) - \frac{1}{2}\sigma^2(t))dt - \frac{u^2}{2}\sigma^2(t)dt + iu\sigma(t)dZ^\theta(t)\right)|\mathcal{F}^X(t)] \\
&= \left(iu(r(t) - \frac{1}{2}\sigma^2(t))dt - \frac{u^2}{2}\sigma^2(t)dt\right)\tilde{\phi}_Y(t)|\mathcal{F}^X(t)(0,t,u). \quad \text{(A.15)}
\end{align*}

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Then it follows that similarly for Equation (A.5), we obtain

$$d\phi_{\tilde{Y}(t)}|\mathcal{F}_X(t)(0, t, u) = \left(-r(t)dt + iu(r(t) - \frac{1}{2}\sigma^2(t))dt - \frac{u^2}{2}\sigma^2(t)dt\right)\phi_{\tilde{Y}(t)}|\mathcal{F}_X(t)(0, t, u). \quad (A.16)$$

Furthermore, we evaluate $dh(t, u)$ via the product rule for stochastic differentiation:

$$dh(t, u) = dX(t)\phi_{\tilde{Y}(t)}|\mathcal{F}_X(t)(0, t, u) + X(t)d\phi_{\tilde{Y}(t)}|\mathcal{F}_X(t)(0, t, u) + dX(t)d\phi_{\tilde{Y}(t)}|\mathcal{F}_X(t)(0, t, u)$$

$$= Ah(t, u)dt + \phi_{\tilde{Y}(t)}|\mathcal{F}_X(t)(0, t, u) + d\phi_{\tilde{Y}(t)}|\mathcal{F}_X(t)(0, t, u) + dM^\theta(t), \quad (A.17)$$

where $diag(g(u))$ is a $N \times N$ diagonal matrix with elements $g_j(u) = -r_j + iu(r_j - \frac{1}{2}\sigma_j^2) - \frac{1}{2}u^2\sigma_j^2$ for $j = 1, 2, ..., N$ along its diagonal. These are the corresponding multiplicative terms from Equation (A.16) under each regime $j$. Then it follows that

$$dE^\theta[h(T, u)] = E^\theta[dh(T, u)]$$

$$= E^\theta\left[\left(A + diag(g(u))\right)h(T, u)dt\right]$$

$$+ E^\theta\left[\phi_{\tilde{Y}(t)}|\mathcal{F}_X(t)(0, T, u) + d\phi_{\tilde{Y}(T)}|\mathcal{F}_X(t)(0, T, u)\right]dM^\theta(t)$$

$$= \left(A + diag(g(u))\right)E^\theta[h(T, u)]dt. \quad (A.18)$$

Integrating with the initial condition $E^\theta[h(0, u)] = X(0)$, gives

$$E^\theta[h(T, u)] = X(0)\exp\left[\left(A + diag(g(u))\right)T\right]. \quad (A.19)$$

Consequently, the discounted characteristic function of the log-return, $\phi_{\tilde{Y}(T)}(0, T, u)$, is given by

$$\phi_{\tilde{Y}(T)}(0, T, u) = \langle E^\theta[h(T, u)], 1 \rangle$$

$$= \left<X(0)\exp\left[\left(A + diag(g(u))\right)T\right], 1 \right>. \quad (A.20)$$

In the case of two regimes, we obtain

$$\phi_{\tilde{Y}(T)}(0, T, u) = \left<X(0)\exp\left[\begin{array}{cc} -a_{12} + g_1(u) & a_{12} \\ a_{21} & -a_{21} + g_2(u) \end{array}\right]T, 1 \right>. \quad (A.21)$$
Appendix B

Derivation of Contract Equations in GMWBs

In this Appendix we derive Equation (3.109) in Section 3.4.2 which governs the annuity value between withdrawals. The method is adopted from Chen et al. (2008), but re-derived under a regime-switching environment as outlined in Equation (3.17).

Consider the following scenario: the underlying asset $W$ in the investor’s account is driven by the regime-switching dynamics

$$dW = (\kappa(t) - \alpha_{tot}) W dt + \sigma(t) W dZ,$$

where $\kappa(t)$ and $\sigma(t)$ are the drift rate and volatility of the fund, respectively, and are described by Equations (3.14) and (3.15); $Z$ represents the Brownian motion under the real-world probability measure. We ignore withdrawals from the account and so $dA = 0$. We also assume that the account is able to track the index $F$ without any basis risk. The index $F$ follows the process:

$$dF = \kappa(t) F dt + \sigma(t) F dZ.$$  

We also assume that it is not possible to short the mutual fund, so that the obvious arbitrage opportunity cannot be exploited, and that an index proxy can be used to hedge the guarantee (Chen et al., 2008). Consider the GMWB value $U(W, A, t)$, and with similar reasoning as in Section 3.3.2, we describe its time-$t$ value conditional on the state of the Markov chain $X$, i.e $U(W, A, t, X(t))$. We set up the hedging portfolio $\Pi(W, F, t, X(t))$ such that

$$\Pi(W, F, t, X(t)) = -U(W, A, t, X(t)) + \chi F,$$
where \( \chi \) is the number of units of the index \( F \). Using Itô’s lemma, and assuming no withdrawal (i.e. between withdrawal dates), we obtain

\[
d\Pi = -dU + \chi dF + \alpha_g W dt
\]

where the term \( \alpha_g W \) represents the GMWB fee collected. We choose

\[
\chi = \frac{W}{F} U W, \tag{B.5}
\]

to remove the \( dZ \) terms so that \( \Pi \) is riskless.

\[
d\Pi = -(U_t dt + U_W dW + \frac{1}{2} U_W W dW^2 + \langle U, d\mathbf{X} \rangle) + \chi dF + \alpha_g W dt, \tag{B.4}
\]

Since \( \Pi \) is now riskless, by no-arbitrage arguments that

\[
d\Pi = r(t) \Pi dt, \] where \( r(t) \) is the risk-free force of interest. Thus, we have

\[
-\left( U_t - \alpha_{tot} U_W + \frac{1}{2} \sigma^2(t) W^2 U_{WW} \right) dt - \sigma(t) U_W W dZ
\]

\[
+ \frac{1}{2} \sigma^2(t) W^2 U_{WW} dt
\]

\[
+ \langle U, d\mathbf{X} \rangle
\]

\[
- \alpha_g W dt + \langle U, \mathbf{A} \mathbf{X} \rangle dt. \tag{B.6}
\]

or

\[
-\left( U_t - \alpha_{tot} U_W + \frac{1}{2} \sigma^2(t) W^2 U_{WW} \right) dt - \sigma(t) U_W W dZ
\]

\[
+ \frac{1}{2} \sigma^2(t) W^2 U_{WW} dt
\]

\[
+ \langle U, \mathbf{A} \mathbf{X} \rangle - \alpha_g W = -U_t \tag{B.7}
\]

or

\[
-\frac{1}{2} \sigma^2(\tau) W^2 U_{WW} \tag{B.8}
\]

where \( \tau = T - t \). Now let us denote \( \mathcal{V}(W, \mathbf{A}, \tau, \mathbf{X}(\tau)) \) as the total variable annuity (including the GMWB rider), that is,

\[
\mathcal{V} = U + W. \tag{B.9}
\]
Then, combining Equation (B.9) with Equation (B.8), we have
\[
V_\tau = -\alpha_g W - r(\tau)(V - W) + W V_W (r(\tau) - \alpha_{tot}) - W (r(\tau) - \alpha_{tot}) + \frac{1}{2} \sigma^2(\tau) W^2 V_{WW} + (V, AX)
\]
\[
= (\alpha_{tot} - \alpha_g) W - r(\tau) V + W V_W (r(\tau) - \alpha_{tot}) + \frac{1}{2} \sigma^2(\tau) W^2 V_{WW} + (V, AX)
\]
\[
= \alpha_m W - r(\tau) V + W V_W (r(\tau) - \alpha_{tot}) + \frac{1}{2} \sigma^2(\tau) W^2 V_{WW} + (V, AX).
\]  

(B.10)

Similar to the derivation of the set of PDEs which govern option prices under the RSLN model in Section 3.3.2, Equation (B.10) is also expanded for all regimes, \( p = 1, \ldots, N \), and we denote \( V^{(p)} = V(W, A, \tau, X(\tau) = e_p) \), i.e. the value of \( V \) under regime \( p \).

\[
V^{(p)}_\tau = (a_{pp} + \tilde{L}^{(p)}) V^{(p)} + \sum_{q \neq p} a_{pq} V^{(q)} + \alpha_m W,
\]

(B.11)

where \( a_{ij} \) is the \((i, j)\)-th component in the transition rate matrix \( A \). Furthermore, \( \tilde{L}^{(p)} \) is the \( \alpha_{tot} \)-adjusted operator such that

\[
\tilde{L}^{(p)} = \frac{1}{2} \sigma_p^2 W^2 \frac{\partial^2}{\partial W^2} + (r_p - \alpha_{tot}) W \frac{\partial}{\partial W} - r_p.
\]

(B.12)

This completes the derivation of the PDEs which govern the value of the GMWB between withdrawal dates.
Appendix C

Zero Padding

In this Appendix, we discuss the wrap-around error brought up in Section 3.4.2. We describe how the error is induced, and the solution we have used to minimise this error.

In the implementation of the FST method we utilise the FFT between time-steps and represent the annuity value $V(W, A, t)$ through a finite discrete grid in the $W$ and $A$ directions, the sub-account and guarantee account. However, applying the FFT causes wrap-around error in the annuity value solution as $V$ is a aperiodic function. This is because when the FFT is applied, $V$ is essentially represented by a periodic function (a series of trigonometric functions) whose period is equal to the grid length. This periodic assumption causes annuity values at the edges of the grid to be “wrapped around” to the other side of the grid. Specifically, wrap-around error occurs at $W >> W_{max}$ and produces inaccurate values at $W \approx W_{max}$. Similarly, this also happens when $W << W_{min}$. The wrap-around effect is especially important in the pricing of GMWB as the maximum condition applied under dynamic withdrawals, Equation (3.112), searches for values across the whole grid. Thus, price inaccuracies near the boundaries of $W$ compromise the values across the whole grid. We are not so worried about the wrap-around effect of $A$ as it takes up a finite set of values $[0, W]$ whereas $W \in [0, \infty)$ which has been truncated into the interval $[W_{min}, W_{max}]$.

Lippa (2013) suggests a simple method to tackle the wrap-around error - zero padding. This method suggests to add zero-value nodes to the computation grid in the $W$ direction. Specifically, if we aim to pad in both directions, we change

$$V'' = [V_1, V_2, ..., V_N]$$

(C.1)
\[ \hat{V}^n = [0, 0, 0, \ldots, V_1, V_2, \ldots, V_N, 0, 0, \ldots, 0], \] (C.2)

where \( \hat{V}^n \) is a \( 2N \times 1 \) vector with \( \frac{N}{2} \) zeros before and after the original \( N \times 1 \) vector \( V^n \). The FFT transform is applied to \( \hat{V} \), and the time-stepping in the FST algorithm is utilised. After the inverse FFT is used, the added zero-value nodes are deleted to retrieve back the vector of size \( N \times 1 \), and this represents \( V^{n+1} \). This method is effective as it doubles the frequency of the FFT which provides more accurate values near the boundaries \( W_{min} \) and \( W_{max} \). This reduces the wrap-around error in the numerical solution for pricing.
Appendix D

Tables

This Appendix provides additional results alluded to in Sections 4.1.4 and 4.2.3. The following tables report the price sensitivity of GMMBs, GMIBs, GMDBs and GMWBs to the regime parameters.
Table D.1: GMMB price sensitivity to regime parameters

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Bibliography


PrudentialUK (2012). Key Features of the Flexible Investment Plan (No Initial Charge and Initial Charge option) - Additional Investments.


